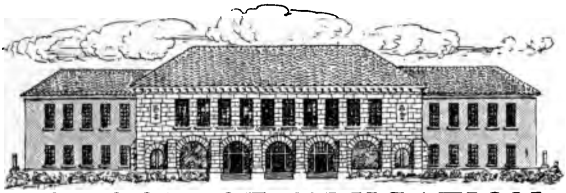

STANDARD ALGEBRA

BY

WILLIAM J. MILNE, Ph.D., LL.D.
PRESIDENT OF NEW YORK STATE NORMAL COLLEGE
ALBANY, N.Y.

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STANDARD ALGEBRA.

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PREFACE

Scope. — This work has been written to meet the requirements of colleges and universities for general admission and of the course outlined by the Regents of the State of New York for both elementary and intermediate algebra. Every kind of question asked in recent examinations has been covered.

Method. — The author adheres to the inductive method of presentation, but uses declarative statements and observations instead of questions. These are followed by illustrative problems and explanations which bring out the important points that should be emphasized, and the treatment is rounded out by abundant practice.

Progress is from the known to the related unknown, and in this way is combined the student's knowledge of arithmetic with the algebraic knowledge to be acquired. New ideas of number are introduced whenever the development of the science demands it.

Exercises. — The number of exercises is extremely large, and the variety is great. The concrete work is well balanced with the abstract, so that both skill in algebraic processes and ability to solve problems are properly sustained.

Problems. — The problems are more distinctly related to real life and business than those found in most algebras. Some of the traditional problems have been retained because they are often given in examinations; and besides, they are useful in developing a sort of intellectual power. But the work contains a large number of fresh and interesting problems drawn from commercial life, from physics and geometry, and from various topics of modern interest. While the formulæ of physics and of geometry are employed to familiarize the pupils with solutions for other letters than x, y, and z, no attempt has been made to present the subject-matter of physics or geometry.

Algebraic Representation. — Throughout the early part of the book there are sets of exercises designed to teach algebraic language. By them the student is required to translate into *algebraic notation* expressions stated in words, and also to state *in words* expressions that are written with algebraic symbols.

Numerical Substitution. — A large amount of work is given in aluating expressions. This is important not only in imparting a better idea of algebraic language, but it is used throught the book in testing results. Accuracy is thus secured by e numerous checks and tests that are suggested and by the quirement that roots of equations be verified. The student this way becomes self-reliant, and reference to answers comes unnecessary.

Graphs. — An interesting sidelight and adjunct in the genal solution of equations is given by the presentation of aphic solutions. They are not to be substituted for the dinary methods of solution; consequently, they are put ter, rather than before, the particular kinds of equations to ich they refer. They will be found to interest the student a phase of algebra which has relation to his more advanced ork in mathematics.

Factoring. — Present-day requirements omit from highest mmon factor and lowest common multiple the method by ccessive division. This makes it imperative that the stunt shall be well prepared in the subject of factoring; conquently, the author has treated not only all the usual cases lly and completely with plenty of practice, but the factor eorem is taught, thus giving the student a method of attackg expressions otherwise very difficult to factor. The summry of cases presented at the close of the chapter on factoring ll give the student unusual power in this important subject. ctoring by completing the square receives attention in the apter on quadratics.

The solution of equations by factoring is treated early in e book, and indeed wherever it is feasible to adopt that ethod.

Reviews. — Helpful and frequent reviews constitute a valuae feature of the Standard Algebra. They call for a knowlge of principles, processes, definitions, and for the solution abstract exercises and of problems.

The main features of the book as specified above will comend it to those who are looking for a text that is thoroughly to date in its matter, clear and intelligible in its presentan, thorough in its method of treatment, and certain to give e student not only a scholarly presentation of the science t delight in its mastery.

WILLIAM J. MILNE.

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INTRODUCTION

1. The basis of algebra is found in arithmetic. Both arithmetic and algebra treat of number, and the student will find in algebra many things that were familiar to him in arithmetic. In fact, there is no clear line of demarcation between arithmetic and algebra. The fundamental principles of each are identical, but in algebra their application is broader than it is in arithmetic.

The very attempt to make these principles universal leads to new kinds of number, and while the signs, symbols, and definitions that are given in arithmetic appear in algebra with their arithmetical meanings, yet in some instances they take on additional meanings.

To illustrate, arithmetic teaches the meaning of $5-3$ and so does algebra, but it will be seen that algebra is more general than arithmetic in that it gives a meaning also to $3-5$, which in arithmetic is meaningless. In this connection the student will see how addition does not always mean an increase, nor subtraction a decrease. Arithmetic teaches the meaning of 9^2; that is, $9^2 = 9 \times 9$. Later the student will learn that algebra gives a meaning to $9^{\frac{1}{2}}$; that is, $9^{\frac{1}{2}} = 3$, one of the two equal factors of 9.

In short, **algebra** *affords a more general discussion of number and its laws than* is found in arithmetic.

ALGEBRAIC SOLUTIONS

2. The numbers in this chapter do not differ in character from the numbers with which the student is already familiar. The following solutions and problems, however, serve to illustrate how the solution of an arithmetical problem may often be made easier and clearer by the *algebraic* method, in which the numbers sought are represented by *letters*, than by the ordinary arithmetical method.

Letters that are used for numbers are called **literal numbers.**

3. Illustrative Problem. — A man had 400 acres of corn and oats. If there were 3 times as many acres of corn as of oats, how many acres were there of each ?

ARITHMETICAL SOLUTION

	A certain number = the number of acres of oats.
Then,	3 times that number = the number of acres of corn,
and	4 times that number = the number of acres of both ;
therefore,	4 times that number = 400.
Hence,	the number = 100, the number of acres of oats,
and	3 times the number = 300, the number of acres of corn.

ALGEBRAIC SOLUTION

Let	$x =$ the number of acres of oats.
Then,	$3x =$ the number of acres of corn,
and	$4x =$ the number of acres of both ;
therefore,	$4x = 400$.
Hence,	$x = 100$, the number of acres of oats,
and	$3x = 300$, the number of acres of corn.

Observe that in the algebraic solution x is used to stand for "a certain number" or "that number," and thus the work is abbreviated.

4. An expression of the equality of two numbers or quantities is called an **equation.**

$5x = 30$ is an equation.

5. A question that can be answered only after a course of reasoning is called a **problem.**

6. The process of finding the result sought is called the **solution** of the problem.

Problems

7. Solve, both arithmetically and algebraically, the following problems:

1. A bicycle and suit cost $54. How much did each cost, if the bicycle cost twice as much as the suit?

2. Two boys dug 160 clams. If one dug 3 times as many as the other, how many did each dig?

3. Two boys bought a boat for $45. One furnished 4 times as much money as the other. How much did each furnish?

4. Find a number whose double equals 52.

5. A certain number added to 3 times itself equals 96. What is the number?

6. The water and steam in a boiler occupied 120 cubic feet of space, and the water occupied twice as much space as the steam. How many cubic feet of space did each occupy?

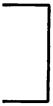
7. A house and lot cost $3000. If the house cost 4 times as much as the lot, what was the cost of each?

8. In a fire B lost twice as much as A, and C lost 3 times as much as A. If their combined loss was $6000, what was the loss of each?

9. A boy bought a bat, a ball, and a glove for $2.25. If the bat cost twice as much as the ball, and the glove cost 3 times as much as the bat, what was the cost of each?

10. A farmer raised a certain number of bushels of wheat, 4 times as much corn, and 3 times as much barley as corn. If there were in all 5100 bushels of grain, how many bushels of each kind did he raise?

. The sides of any square (Fig. 1) are equal in length. long is one side of a square, if the perimeter (distance nd it) is 36 inches?



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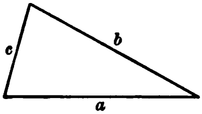


FIG. 2

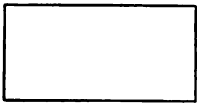


FIG. 3

. The length of each of the sides, *a* and *b*, of the triangle 2) is twice the length of the side *c*. If the perimeter is .ches, what is the length of each side?

. The opposite sides of any rectangle (Fig. 3) are equal. rectangle is twice as long as it is wide and its perimeter inches, how wide is it? How long?

. In a business enterprise the joint capital of A, B, and C $8400. If A's capital was twice B's, and B's was twice what was the capital of each?

. The owner of a piano found that the annual cost of ing it in tune and insuring it against fire was $12.50, and the cost of keeping it in tune was 9 times the cost of :ing it. Find the cost of each item.

. One year 1500 violins were made in the United States. e as many were made in New York as in Massachusetts, these two states made half of all that were made in the ed States. How many violins were made in Massa- etts? in New York?

. Messrs. Jones, Hollis & Frye invested $225,000 in a of steamboats. Mr. Hollis invested 3 times as much as Jones, and Mr. Frye 5 times as much as Mr. Jones. How ı did each invest?

A plumber and two helpers together earned $7.50 per How much did each earn per day, if the plumber earned *s as much as each* helper?

19. Divide 21 into three parts, such that the first is twice the second, and the second is twice the third.

20. Divide 36 into three parts, such that the first is twice the second, and the third is twice the sum of the first two.

21. Three newsboys sold 60 papers. If the first sold twice as many as the second, and the third sold 3 times as many as the second, how many did each sell?

22. Henry earned a certain number of dollars per day. With 5 days' earnings he purchased a rifle, and with 20 days' earnings, a bicycle. If both together cost $50, how much did he earn per day? How much did the rifle cost? the bicycle?

23. A man sold some ducks for 50 cents each, and the same number of geese for 75 cents each. If he received $12.50 for all, how many of each did he sell?

24. A and B began business with a capital of $7500. If A furnished half as much capital as B, how much did each furnish?

SUGGESTION. — Let $x =$ the number of dollars A furnished.

25. James bought a pony and a saddle for $60. If the saddle cost $\frac{1}{3}$ as much as the pony, find the cost of each.

26. Separate 72 into two parts, one of which shall be $\frac{1}{3}$ of the other.

27. Separate 78 into two parts, one of which shall be $\frac{1}{5}$ of the other.

28. A basket-ball team won 16 games, or $\frac{2}{3}$ of the games it played. Find the number of games it played.

SOLUTION

Let $x =$ the number of games it played.

Then, $\frac{2}{3}x = 16$,

and $\frac{1}{3}x = 8$.

Therefore, $x = 24$, the number of games it played.

29. The distance by rail between two cities is 35 miles. This is $\frac{5}{6}$ of the distance by boat. Find the distance by boat.

30. The United States sent to Germany one year 135,000 pairs of shoes. This was $\frac{3}{5}$ of the number sent the following year. How many pairs of shoes were exported to Germany the second year?

31. If $\frac{3}{8}$ of the number of persons who went on an excursion to Niagara Falls were teachers, and 240 teachers went, find the whole number of persons that went.

32. Find the number of feet in the width of a street, if $\frac{3}{5}$ of the width, or 48 feet, lies between the curbstones.

33. During a mild February, coke declined $\frac{3}{14}$ in market price. The price at the end of the month was $2.20 per ton. What was the price at the beginning of the month?

34. On an elevated belt-line railroad, $9\frac{3}{5}$ minutes, or $\frac{4}{15}$ of the time required to make a round trip, was consumed in stops. Find the number of minutes required to make a round trip.

35. If $\frac{1}{5}$ of a number is added to the number, the sum is 12. What is the number?

36. If $\frac{1}{3}$ of a number is added to twice the number, the sum is 35. What is the number?

37. The difference between $\frac{2}{3}$ of a certain number and $\frac{2}{5}$ of it is 16. What is the number?

38. The number 150 can be divided into two parts, one of which is $\frac{3}{7}$ of the other. What are the parts?

39. I owe in all $93 to A, B, and C. If I owe A $\frac{2}{3}$ as much as C, and B $\frac{2}{5}$ as much as C, how much do I owe each?

40. For every car load of iron ore dumped into a furnace, $\frac{7}{8}$ of a car load of coke was used for fuel and $\frac{3}{8}$ of a car load of limestone was used for a flux. In all 450 car loads of ore, coke, and limestone were used per day in the furnace. How much of each was used per day?

DEFINITIONS AND NOTATION

8. A unit or an aggregate of units is called a **whole number,** or an **integer**; one of the equal parts of a unit or an aggregate of equal parts of a unit is called a **fractional number.**

Such numbers are called *arithmetical,* or *absolute, numbers.*

9. Arithmetical numbers have fixed and known values, and are represented by symbols called **numerals**; as the Arabic *figures,* 1, 2, 3, etc., and the Roman *letters,* I, V, X, etc.

10. You have seen that it is convenient, in solving problems, to use letters for the numbers whose values are sought. So also, in stating rules, letters are used to represent not only the numbers whose values are to be *found,* but also the numbers that must be *given* whenever the rule is applied.

For example, the volume of any rectangular prism is equal to the area of the base multiplied by the height. By using V for volume, A for area of base, and h for height, this rule is stated in symbols, thus:

$$V = A \times h.$$

When $A = 60$ and $h = 5$, $\quad V = 60 \times 5 = 300$;

when $A = 36$ and $h = 10$, $\quad V = 36 \times 10 = 360$; etc.

An equation that states a rule in brief form is called a **formula.**

In each particular problem to which the above formula applies, A and h represent *fixed, known* values, but in consequence of being used for all problems of this class, A and h *represent numbers to which any arithmetical values whatever may be assigned.*

11. A literal number to which any value may be assigned at pleasure is called a **general number.**

12. A general number or a number whose value is known is called a **known number.**

The general numbers, A and h, in the formula $V = A \times h$ (§ 10), are known numbers; so also are the numerals 3 and 5.

13. A number whose value is to be found is called an **unknown number.**

In $3x = 21$, x is an unknown number; in the formula for volume, $V = A \times h$, V is an unknown number; but when this formula is changed to the formula for height, $h = V \div A$, V and A are known numbers and h is an unknown number.

ALGEBRAIC SIGNS

14. The **sign of addition** is $+$, read '*plus.*'

It indicates that the number following it is to be added to the number preceding it.

$a + b$, read 'a plus b,' means that b is to be added to a.

15. The **sign of subtraction** is $-$, read '*minus.*'

It indicates that the number following it is to be subtracted from the number preceding it.

$a - b$, read 'a minus b,' means that b is to be subtracted from a.

16. The **sign of multiplication** is $\times$ or the dot $(\cdot)$, read '*multiplied by.*'

It indicates that the number preceding it is to be multiplied by the number following it.

$a \times b$, or $a \cdot b$, means that a is to be multiplied by b.

The sign of multiplication is usually omitted in algebra, except between figures.

Instead of $a \times b$, or $a \cdot b$, usually ab is used. But 3×5 cannot be written 35, because 35 means $30 + 5$.

17. The **sign of division** is $\div$, read '*divided by.*'

It indicates that the number preceding it is to be divided by the number following it.

$a \div b$ means that a is to be divided by b.

Division may be indicated also by writing the dividend above the divisor with a line between them.

Such indicated divisions are called **fractions**.

$\frac{a}{b}$, sometimes read '*a* over *b*,' means that *a* is to be divided by *b*.

18. The **sign of equality** is $=$, read '*is equal to*' or '*equals*.'

19. **Order of operations**. — It has become a matter of agreement, or a custom, among mathematicians to employ signs of operation, when written in a sequence, as follows:

When only $+$ and $-$ occur in a sequence or only $\times$ and $\div$, the operations are performed in order from left to right.

Thus, $3 + 4 - 2 + 3 = 7 - 2 + 3 = 5 + 3 = 8$;

also, $3 \times 4 \div 2 \times 3 = 12 \div 2 \times 3 = 6 \times 3 = 18$.

$a + b - c + d$ means that b is to be added to a, then from this result c is to be subtracted, and to the result just obtained d is to be added.

When $\times$, $\div$, or both, occur in connection with $+$, $-$, or both, the indicated multiplications and divisions are performed first unless otherwise indicated.

Thus, $7 + 10 - 6 \div 3 \times 4 = 7 + 10 - 2 \times 4 = 7 + 10 - 8 = 9$.

There are *apparent* exceptions to the established order.

For example, in $m \div ab$ the multiplication is considered as already performed; consequently, $m \div ab$ means that m is to be divided by ab, not that m is to be divided by a and the result multiplied by b; but $m \div a \times b$ (the multiplication being indicated by $\times$) means that m is to be divided by a and the result multiplied by b.

20. The **signs of aggregation** are: the *parentheses*, ( ); the *vinculum*, $\overline{}$; the *brackets*, [ ]; the *braces*, { }; and the *vertical bar*, |.

They are used to group numbers, each group being regarded as a single number.

Thus, each of the forms $(a + b)c$, $\overline{a + b} \cdot c$, $[a + b]c$, $\{a + b\}c$, and $\begin{array}{r|l} a & c \\ +\,b & \end{array}$ signifies that the sum of a and b is to be multiplied by c.

All operations within groups should be performed first.

When numbers are included by any of the signs of aggregation, they are commonly said to be in parenthesis, in a parenthesis, or in parentheses.

21. The **sign of continuation** is $\cdots$, read '*and so on,*' or '*and so on to.*'

2, 4, 6, 8, $\cdots$, 50 is read '2, 4, 6, 8, and so on to 50.'

22. The **sign of deduction** is $\therefore$, read '*therefore*' or '*hence.*'

EXERCISES

23. Read and tell the meaning of:

1. $m+n$.
2. $x-y$.
3. $a \div b$.
4. $\frac{a}{b}+\frac{r}{s}$.
5. $ab-rs$.
6. $2 \cdot 3-4w$.
7. $3p+5q$.
8. $7(v+z)$.
9. $\frac{a-r}{b+s}$.
10. $\overline{x+y} \div 3$.
11. $a+m+n$.
12. $a+(m-n)$.
13. $a-\overline{m-n}$.
14. $\frac{1}{a}-\frac{2}{m}+\frac{3}{n}$.
15. $(a+m)(b-n)$.

Indicate results:

16. Add 2 times c to 5 times d.

17. Subtract 2 times 4 from m times n.

18. Multiply the sum of x and y by z.

19. Divide $v-w$ by r times s.

20. Find the product of $2x+7$ and $3y-2$.

21. Express the product of a and $a+b$ divided by the product of b and $a-b$.

22. A boy had a apples and his brother gave him b more. How many apples had he then?

23. Edith is 14 years old. How old was she 4 years ago? a years ago? How old will she be in 3 years? in b years?

24. At x cents each, how much will 5 oranges cost?

25. If z caps cost 10 dollars, how much will 1 cap cost?

26. At y cents each, how many pencils can be bought for x cents?

27. A boy who has p marbles loses q marbles, and afterward uys r marbles. How many marbles does he then have?

28. What two whole numbers are nearest to 9? to x, if x is a whole number? to a, if a is a whole number?

29. If y is an even number, what are the two nearest even numbers?

30. A woman exchanged x dozen eggs for 8 pounds of sugar at a cents a pound and 5 pounds of coffee at b cents a pound. How much were the eggs worth a dozen?

FACTORS, POWERS, AND ROOTS

24. Each of two or more numbers whose product is a given number is called a **factor** of the given number.

Since $12 = 2 \times 6$, or 4×3, each of these numbers is a factor of 12.

Since $3ab = 3 \times a \times b$, or $3a \times b$, or $3 \times ab$, or $3b \times a$, each of these numbers, 3, a, b, $3a$, $3b$, and ab, is a factor of $3ab$.

25. When one of the two factors into which a number can be resolved is a *known* number, it is usually written first and called the **coefficient** of the other factor.

In $5xy$, 5 is the coefficient of xy; in ax, if a is a known number, it is the coefficient of x.

In a broader sense, either one of the two factors into which a number can be resolved may be considered the *coefficient*, or *co-factor*, of the other.

In $5ax$, ax may be considered the coefficient of 5, $5a$ of x, x of $5a$, etc.

Coefficients are **numerical**, **literal**, or **mixed**, according as they are composed of *figures*, *letters*, or both *figures and letters*.

When no numerical coefficient is expressed, the coefficient is considered to be 1.

26. When a number is used a certain number of times as a factor, the product is called a **power** of the number.

When a is used *twice* as a factor, the product is the **second power** of a, or the **square** of a; when a is used *three* times as a factor, the product is the **third power** of a, or the **cube** of a; *four times, the* **fourth power** of a; n times, that is, any number *of times, the* **nth power** of a.

27. The product indicated by $a \times a \times a \times a \times a$ may be indicated more briefly by a^5. Likewise, if a is to be used n times as a factor, the product may be indicated by a^n.

A figure or a letter placed a little above and to the right of a number is called an **index**, or an **exponent**, of the power thus indicated.

The exponent indicates how many times the number is to be used as a factor.

5^2 indicates that 5 is to be used twice as a factor; a^3 indicates that a is to be used 3 times as a factor.

a^2 is read 'a square,' or 'a second power'; a^3 is read 'a cube,' or 'a third power'; a^4 is read 'a fourth,' 'a fourth power,' or 'a exponent 4'; a^n is read 'a nth,' 'a nth power,' or 'a exponent n.'

When no exponent is written, the exponent is regarded as 1.

5 is regarded as the first power of 5, and a^1 is usually written a.

The terms *coefficient* and *exponent* should be distinguished.

$5a$ means $a + a + a + a + a$, but a^5 means $a \times a \times a \times a \times a$.

28. When the factors of a number are all equal, one of the factors is called a **root** of the number.

5 is a root of 25; a is a root of a^4; $4x$ is a root of $64x^3$.

One of the *two* equal factors of a number is its **second**, or **square, root**; one of the *three* equal factors of a number is its **third**, or **cube, root**; one of the *four* equal factors, the **fourth root**; one of the n equal factors, the **nth root**.

29. The symbol which denotes that a root of a number is sought is $\sqrt{}$, written before the number.

It is called the **root sign**, or the **radical sign**.

A figure or a letter written in the opening of the radical sign indicates what root of the number is sought.

It is called the **index** of the root.

When no index is written, the second, or square, root is meant.

$\sqrt[3]{8}$ indicates that the third, or cube, root of 8 is sought.

$\sqrt{ax}$ indicates the square root of ax, and $\sqrt{a-b}$, the square root of $a-b$.

ALGEBRAIC EXPRESSIONS

30. A number represented by algebraic symbols is called an **algebraic expression.**

31. An algebraic expression whose parts are not separated by + or − is called a **term**; as $2x^2$, $-5xyz$, and $\frac{xy}{z}$.

In the expression $2x^2 - 5xyz + \frac{xy}{z}$ there are three terms.

The expression $m(a+b)$ is a term, the parts being m and $(a+b)$.

32. Terms that contain the same letters with the same exponents are called **similar terms.**

$3x^2$ and $12x^2$ are similar terms; also $3(a+b)^2$ and $12(a+b)^2$; also ax and bx, when a and b are regarded as the coefficients of x.

33. Terms that contain different letters, or the same letters with different exponents, are called **dissimilar terms.**

$5a$ and $3by$ are dissimilar terms; also $3a^2b$ and $3ab^2$.

34. An algebraic expression of one term only is called a **monomial,** or a **simple expression.**

xy and $3ab$ are monomials.

35. An algebraic expression of more than one term is called a **polynomial,** or a **compound expression.**

$3a + 2b$, $xy + yz + zx$, and $a^2 + b^2 - c^2 + 2ab$ are polynomials.

36. A polynomial of two terms is called a **binomial.**

$3a + 2b$ and $x^2 - y^2$ are binomials.

37. A polynomial of three terms is called a **trinomial.**

$a + b + c$ and $3x - 2y - z$ are trinomials.

38. An expression, any term of which is a fraction, is called a **fractional expression.**

$\frac{2x^2}{a^2} - 3x + \frac{a}{x}$ is a fractional expression.

39. An expression that contains no fraction is called an **integral expression.**

$5a^2 - 2a$ and $6x$ are integral expressions.

Expressions like $x^3 + \frac{3}{4}x^2 + \frac{1}{2}x + 1$ are sometimes regarded as integral, since the literal numbers are not in fractional form.

Numerical Substitution

40. When a particular number takes the place of a letter or general number, the process is called **substitution.**

EXERCISES

41. 1. When $a = 2$ and $b = 3$, find the numerical value of $3ab$; of a^4.

SOLUTIONS. $3ab = 3 \cdot 2 \cdot 3 = 18$; also, $a^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16$.

When $a = 5$, $b = 3$, $c = 10$, $m = 4$, find the value of:

2. $10a$.	6. $5m^2$.	10. am^4.	14. $\frac{1}{3}ab^2$.
3. $2ab$.	7. $2a^2b$.	11. $(ab)^2$.	15. $\frac{1}{2}bm$.
4. $3cm$.	8. $3bm^3$.	12. a^2b^2.	16. $\frac{1}{5}abc$.
5. $6bc$.	9. $4a^3b$.	13. a^bc.	17. $3b^2cm^2$.

18. When $x = 6$, $y = 3$, $z = 2$, $m = 0$, $n = 4$, find the value of $\sqrt{xyz^3}$; of $3m^2n$.

SOLUTIONS

$$\sqrt{xyz^3} = \sqrt{6 \cdot 3 \cdot 2 \cdot 2 \cdot 2} = \sqrt{144} = 12.$$
$$3m^2n = 3 \cdot 0^2 \cdot 4 = 3 \cdot 0 \cdot 4 = 0.$$

NOTE. — It will be seen in § 541 that when any factor of a product is 0, the product is 0; therefore, any power of 0 is 0; also any root of 0 is 0.

When $a = 4$, $b = 2$, $r = 0$, $s = 5$, find the value of:

19. $\sqrt{2ab}$.	22. $\sqrt{7r^3s}$.	25. $3a\sqrt{b^2s^4}$.	28. $6^b\sqrt{s^2b^a}$.
20. $7b^2r$.	23. $3s^bb^a$.	26. $\frac{3}{8}a^3bs$.	29. $2^ab^3s^2r^4$.
21. $\sqrt{as^2}$.	24. $\sqrt[3]{8ab}$.	27. $.8s\sqrt{a^2b^4}$.	30. $\sqrt[3]{9^ar^sb^6}$.

31. $\dfrac{3a^2b}{sb}$. 32. $\dfrac{bs^4r}{abs}$. 33. $\dfrac{6a^sb^a}{b^{2s}a^4}$. 34. $\dfrac{24a^bb^ss^a}{6a^3b^3s^3}$.

35. When $x=3$ and $y=2$, find the value of x^2-y^2; of $(x-y)^2$; of $x^2-2xy+y^2$.

SOLUTIONS

$$x^2 - y^2 = 3 \cdot 3 - 2 \cdot 2 = 9 - 4 = 5.$$
$$(x - y)^2 = (3 - 2)^2 = 1 \cdot 1 = 1.$$
$$x^2 - 2xy + y^2 = 3 \cdot 3 - 2 \cdot 3 \cdot 2 + 2 \cdot 2 = 9 - 12 + 4 = 1.$$

When $a=5$, $b=3$, $m=4$, $n=1$, find the value of:

36. a^2+b^2.

37. $(a+b)^2$.

38. $\dfrac{m+2n}{m-2n}$.

39. $(n-1)^5$.

40. n^5-1.

41. $m+\dfrac{2n}{m-2n}$.

42. m^{a-b}.

43. $(bm)^{a-b}$.

44. $\dfrac{a^3-b^3}{a-b}$.

45. $ab - bn + mb^2 \div 3mn^2$.

46. $(ab - bn + mb^2) \div 3mn^2$.

47. $2^am^2n^2 - abmn \div 4bn - m^3n^7$.

48. $\frac{1}{2}m + 3a^2b - \frac{2}{3}b^2m^2 - 8a$.

49. $\frac{3}{5}a^2m + \frac{1}{2}m^2n - \frac{2}{9}ab^3 - n^4$.

50. $ambn^2 - \frac{3}{4}b^2m + \frac{5}{8}m^2n^3 - \frac{1}{5}m^3$.

51. Show that $2x+3x=5x$ when $x=2$; when $x=3$. Giving x any value you choose, find whether $2x+3x=5x$.

52. Show that $m(a+b)=ma+mb$ when $m=5$, $a=4$, and $b=3$. Find whether the same relation holds true for other values of m, a, and b.

53. Show that $(a+b)^2=a^2+2ab+b^2$ when $a=3$ and $b=2$. Find whether this is true for other values of a and b.

54. When $a=0$, $b=8$, $c=5$, $d=3$, find the value of $c^2d + abc - bd^2 - \frac{3}{4}ad^4$.

SOLUTION

$$c^2d + abc - bd^2 - \tfrac{3}{4}ad^4 = 5 \cdot 5 \cdot 3 + 0 \cdot 8 \cdot 5 - 8 \cdot 3 \cdot 3 - \tfrac{3}{4} \cdot 0 \cdot 3 \cdot 3 \cdot 3 \cdot 3$$
$$= 75 + 0 - 72 - 0 = 3.$$

When $x = 6$, $y = 3$, $z = 0$, $r = 2$, $s = 10$, find the value of:

55. $rx + yz + rs - xz$.

56. $sx^2 - r^2s + xyz - xy^2$.

57. $12z^3 + r^3y^2 + 5xy \div 3s$.

58. $\frac{1}{3}xr^2 - \frac{1}{2}y^4z + \frac{3}{5}s^2$.

59. $4^r sy^2 \div \frac{5}{8}x^2r^2 - \frac{1}{1\frac{1}{2}}x^3y^2z$.

60. $5xy^4 - y\sqrt{r^2s^2} + \frac{5}{6}xsz$.

61. $x^2 + y^2 + z^2 - 2xy + 2xz - 2yz$.

62. $(x - y)^2 + 2(x - y)(r + s) + (r + s)^2$.

63. $5x + 3rs \div 2x \times 7y + 14x^r + \sqrt{xyz}$.

42. Solution of problems in physics by substitution in formulae.
(*a*) If an automobile goes 30 feet per second, in 25 seconds it will go 25 times 30 feet.

In general, the space (s) passed over by anything moving with uniform velocity (v) in any given time (t) is equal to the product of the velocity and the time.

This **physical law** is expressed briefly by the **algebraic formula,**

$$s = vt.$$

By substitution in this formula find the space passed over:

1. By a train in 30 sec., uniform velocity 48 ft. per sec.

2. By a launch in 11 hr., uniform velocity 8.2 mi. per hr.

3. By a torpedo in 75 sec., uniform velocity 44 ft. per sec.

(*b*) The formula for the space (s) through which a freely falling body acted upon by gravity (g) will fall in t seconds, starting from rest, is

$$s = \frac{1}{2}gt^2.$$

Use 32.16 or 9.8 for g according as s is to be obtained in feet or in meters.

4. How many feet will a body fall from rest in 5 seconds?

5. A stone dropped from the top of an overhanging cliff reached the bottom in 4 seconds. How many feet high was the cliff? How many meters high?

6. A ball thrown vertically into the air returned in 6 seconds. How many meters high was it thrown? (Time of fall $= 3$ sec.)

POSITIVE AND NEGATIVE NUMBERS

43. For convenience, arithmetical numbers may be arranged in an ascending scale:

$$0, \quad 1, \quad 2, \quad 3, \quad 4, \quad 5, \quad \cdots$$

The operations of addition and subtraction are thus reduced to counting along a scale of numbers. 2 is added to 3 by beginning at 3 in the scale and counting 2 units in the ascending, or *additive,* direction; and consequently, 2 is subtracted from 3 by beginning at 3 and counting 2 units in the descending, or *subtractive,* direction. In the same way 3 is subtracted from 3. But if we attempt to subtract 4 from 3, we discover that the operation of subtraction is restricted in arithmetic, inasmuch as a greater number cannot be subtracted from a less. If this restriction held in algebra, it would be impossible to subtract one literal number from another without taking into account their arithmetical values. Therefore, this restriction must be removed in order to proceed with the general discussion of numbers.

To subtract 4 from 3 we begin at 3 and count 4 units in the descending direction, arriving at 1 on the opposite, or subtractive, side of 0. It now becomes necessary to extend the scale 1 unit in the subtractive direction from 0.

To subtract 5 from 3 we begin at 3 and count 5 units in the descending direction, arriving at 2 on the opposite, or subtractive, side of 0. The scale is again extended; and in a similar *way the scale* may be extended indefinitely in the *subtractive direction.*

Numbers on opposite sides of 0 may be distinguished by means of the small signs $^+$ and $^-$, called *signs of quality*, or *direction signs*, $^+$ being prefixed to those numbers which stand in the *additive* direction from 0 and $^-$ to those which stand in the *subtractive* direction from 0.

The former are called **positive numbers**, the latter **negative numbers.**

Zero is defined as the result of subtracting any number from itself. Zero is neither positive nor negative.

Including zero, the *scale* of *algebraic numbers* may be written:

$$\ldots,\ {}^-5,\ {}^-4,\ {}^-3,\ {}^-2,\ {}^-1,\ 0,\ {}^+1,\ {}^+2,\ {}^+3,\ {}^+4,\ {}^+5,\ \ldots$$

44. By repeating $^+1$ as a unit any positive integer may be obtained, and by repeating $^-1$ as a unit any negative integer may be obtained. Hence, positive integers are measured by the **positive unit,** $^+1$, and negative integers by the **negative unit,** $^-1$.

Fractions are measured by positive or negative *fractional* units. Thus, the unit of $^+(\frac{7}{8})$ is $^+(\frac{1}{8})$; the unit of $^-(\frac{7}{8})$ is $^-(\frac{1}{8})$.

45. If $^+1$ and $^-1$, or $^+2$ and $^-2$, or any two numbers numerically equal but opposite in quality are taken together, they cancel each other. For counting any number of units from 0 in either direction and then counting an equal number of units from the result in the opposite direction, we arrive at 0.

Hence, *if a positive and a negative number are united into one number, any number of the units or parts of units of which one is composed cancels an equal number of units or parts of units of the other.*

46. Quantities opposed to each other in such a way that, if united, any number of units of one cancels an equal number of the other may be distinguished as **positive** and **negative.**

Thus, if money *gained* is *positive*, money *lost* is *negative*; if a *rise* in *temperature is positive*, a *fall* in temperature is *negative*; if *west* longitude *is positive*, *east* longitude is *negative*; etc.

47. Positive and negative numbers, whether integers or fractions, are called **algebraic numbers.**

Arithmetical numbers are positive numbers.

48. The value of a number without regard to its sign is called its **absolute value.**

Thus, the absolute value of both $+4$ and -4 is 4.

ADDITION AND SUBTRACTION

49. The aggregate value of two or more algebraic numbers is called their **algebraic sum**; the numbers are called **addends.**

The process of finding a simple expression for the algebraic sum of two or more numbers is called **addition.**

50. In addition, two numbers are given, and their algebraic sum is required; in subtraction, the algebraic sum, called the **minuend,** and one of the numbers, called the **subtrahend,** are given, and the other number, called the **remainder,** or **difference,** is required. **Subtraction** is, therefore, the **inverse of addition.**

The **difference** is the algebraic number that *added to the subtrahend gives the minuend.*

51. Negative numbers give the foregoing definitions a wider range of meaning than they had in arithmetic. In algebra addition does not always imply an increase, nor subtraction a decrease.

Sum of Two or More Numbers

52. Add:

EXERCISES

1.	$^+5$	$^-5$	$^+5$	$^+5$	$^+5$	$^-5$	$^-5$
	$^+2$	$^-2$	$^-5$	$^-4$	$^-9$	$^+8$	$^+2$

SUGGESTIONS. — The sum of 2 positive units and 5 positive units is 7 positive units; of 2 negative units and 5 negative units, 7 negative units; (§ 45) *of 5 negative* units and 5 positive units, 0; of 4 negative units *and 5 positive units,* 1 positive unit; of $^-9$ and $^+5$, $^-4$; etc.

To add two algebraic numbers:

RULE. — *If they have* **like signs,** *add the absolute values and prefix the common sign; if they have* **unlike signs,** *find the difference of the absolute values and prefix the sign of the numerically greater.*

By successive applications of the above rule any number of numbers may be added.

Add:

2.	$^{+}8$	$^{+}11$	$^{-}10$	$^{+}12$	$^{-}16$	$^{-}20$
	$^{+}2$	$^{-}7$	$^{+}4$	$^{+}8$	$^{-}4$	$^{+}5$

3.	$^{+}7$	$^{-}5$	$^{+}8$	$^{+}6$	$^{-}9$	$^{+}10$
	$^{-}3$	$^{-}3$	$^{-}9$	$^{+}5$	$^{+}3$	$^{-}40$
	$^{+}2$	$^{-}8$	$^{+}1$	$^{-}5$	$^{+}2$	$^{+}25$

4. $^{+}10 + {}^{-}4 + {}^{-}6 + {}^{-}7 + {}^{+}9.$
5. $^{-}12 + {}^{+}8 + {}^{+}2 + {}^{-}6 + {}^{+}2.$
6. $^{-}40 + {}^{-}6 + {}^{+}8 + {}^{+}7 + {}^{+}6.$
7. $^{+}6 + {}^{-}9 + {}^{+}5 + {}^{+}3 + {}^{-}4.$
8. $0 + {}^{-}7 + {}^{+}4 + {}^{-}3 + {}^{-}4.$
9. $^{-}2 + {}^{+}3 + {}^{+}6 + {}^{+}8 + {}^{-}8.$

53. Abbreviated notation for addition.

Reference to the scale of algebraic numbers shows that adding positive units to any number is equivalent to counting them in the positive direction from that number, and adding negative units to any number is equivalent to counting them in the negative direction from that number. Hence, in addition, the signs + and − denoting quality have primarily the same meanings as the signs + and − denoting arithmetical addition and subtraction. For example,

$$^{+}1 \text{ means } 0 + 1 \text{ and } {}^{-}1 \text{ means } 0 - 1;$$

also $$^{+}5 \text{ means } 0 + 5 \text{ and } {}^{-}5 \text{ means } 0 - 5; \text{ etc.}$$

Hence, in finding the sum of any given numbers, only one set of signs, + and −, is necessary, and they may be regarded either as signs of quality or as signs of operation, though commonly it is preferable to regard them as signs of operation.

Thus, $^{+}5 + {}^{+}3 + {}^{-}6$ may be written $+ 5 + 3 - 6$, or $5 + 3 - 6$.

When the first term of an expression is positive, the sign is usually omitted, as in the preceding illustration.

If there is need of distinguishing between the signs of quality + and − and the signs of operation + and −, the numbers and their signs of quality may be inclosed in parentheses.

Thus, if $a = 5$, $b = -3$, and $c = -2$, then $a + b + c = 5 + (-3) + (-2)$; $a - b - c = 5 - (-3) - (-2)$; $abc = 5(-3)(-2)$; etc.

54. A term preceded by +, expressed or understood, is called a **positive term**, and a term preceded by −, a **negative term**.

Thus, in the polynomial $3 + 2b - 5c$ the first and second terms are positive and the third term is negative.

EXERCISES

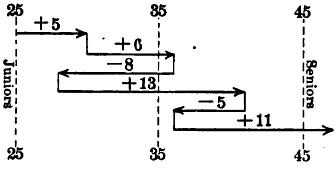
55. Write with one set of signs and find the sum:

1. $^+7 + {^+8}$.
2. $^+6 + {^-5}$.
3. $^-3 + {^-7}$.
4. $^+2 + {^-4}$.
5. $^-6 + {^-3} + {^+16}$.
6. $^+8 + {^+9} + {^-15}$.

Find the sum:

7. $10 - 7 + 4 - 9 - 6 + 3 + 5 - 16 + 24 - 11$.
8. $21 + 3 - 6 - 5 + 8 - 7 + 4 + 12 - 30 + 15$.
9. $17 - 2 - 3 - 4 - 6 + 8 - 2 + 40 - 18 + 13$.

10. In a football game the ball was advanced 5 yards from the Juniors' 25-yard line toward the Seniors' goal, then 6 yards, then −8 yards (*i.e.* it went back 8 yards), and so on, as shown in the diagram.



What was the position of the ball after 3 plays? after 4 plays? after 5 plays? after 6 plays?

11. Plot the following and find the last position of the ball:

On 15-yard line; gained 4 yards; gained 5 yards; lost 2 yards; *gained 30 yards;* lost 6 yards; lost 2 yards; gained 12 yards.

Difference of Two Numbers

EXERCISES

56. On account of the extension of the scale of numbers below zero (§ 43), subtraction is always possible in algebra.

When the subtrahend is positive, algebraic subtraction is like arithmetical subtraction, and consists in counting *backward* along the scale of numbers.

Subtract the lower number from the upper one:

1.	6	6	6	6	6	6	6
	3	4	5	6	7	8	9
2.	${}^-3$	${}^-3$	${}^-3$	${}^-4$	${}^-5$	${}^-6$	${}^-7$
	0	1	2	3	4	5	6

Observe that *subtracting a positive number is equivalent to adding a numerically equal negative number.*

When the subtrahend is negative, it is no longer possible to subtract as in arithmetic by counting backward.

3. Subtract ${}^-2$ from 8.

PROCESS

$$\begin{array}{r} 8 \\ -2 \\ \hline 8+2=10 \end{array}$$

EXPLANATION. — If 0 were subtracted from 8, the result would be 8, the minuend itself.

The subtrahend, however, is not 0, but is a number 2 units below 0 in the scale of numbers. Hence, the difference is not 8, but is $8 + 2$, or the minuend plus the subtrahend with its sign changed.

Subtract the lower number from the upper one:

4.	4	4	4	4	5	7	9
	0	${}^-1$	${}^-2$	${}^-3$	${}^-6$	${}^-7$	${}^-5$
5.	${}^-5$	${}^-5$	${}^-5$	${}^-5$	${}^-1$	${}^-4$	${}^-6$
	0	${}^-1$	${}^-2$	${}^-6$	${}^-3$	${}^-7$	${}^-5$

PRINCIPLE. — *Subtracting any number is equivalent to adding it with its sign changed.*

Subtract the lower number from the upper one:

6.	10	12	20	16	40	32
	$^{-}2$	$^{-}5$	$^{-}6$	$^{-}4$	$^{-}8$	$^{-}7$
7.	0	$^{-}3$	$^{-}7$	$^{-}10$	$^{-}5$	$^{-}12$
	$^{-}2$	$^{-}6$	$^{-}4$	$^{-}5$	$^{-}10$	$^{-}20$
8.	4	4	$^{-}4$	$^{-}9$	3	$^{-}7$
	4	$^{-}4$	4	3	$^{-}9$	8
9.	5	$^{-}3$	$^{-}5$	10	$^{-}10$	13
	$^{-}6$	7	$^{-}4$	$^{-}7$	7	$^{-}9$
10.	20	44	28	$^{-}10$	$^{-}10$	$^{-}5$
	3	$^{-}4$	$^{-}6$	10	$^{-}10$	12

11. Subtract 12 from $^{-}1$.

12. Subtract $^{-}4$ from 14.

13. Subtract 11 from $^{-}8$.

14. From $^{-}6$ subtract 4.

15. From 0 subtract $^{-}3$.

16. From $^{-}3$ subtract 0.

17. From 0 subtract $^{-}7$; from the result subtract $^{-}4$; then add $^{-}2$; add $^{-}3$; add 7; subtract 11; and add $^{-}6$.

A weather map for January 16 gave the following minimum and maximum temperatures (Fahrenheit):

	Chicago	Duluth	Helena	Montreal	New Orleans	New York
Minimum	24°	$^{-}6°$	$^{-}12°$	$^{-}12°$	64°	20°
Maximum	30°	2°	$^{-}4°$	18°	76°	42°

18. The range of temperature in Chicago was 6°. Find the range of temperature in each of the other cities.

19. The freezing point is 32° F. How far below the freezing point did the temperature fall in Montreal?

20. How much colder was it in Duluth than in Chicago? in Montreal than in New York? in Helena than in New Orleans?

ADDITION

57. Arithmetically, $2+3=3+2$.

In general, $a+b=b+a$. That is,

Numbers may be added in any order.

This is the **law of order**, or the **commutative law**, for addition.

58. Arithmetically,

$$2+3+5=(2+3)+5=2+(3+5)=(2+5)+3.$$

In general,

$$a+b+c=(a+b)+c=a+(b+c)=(a+c)+b.$$ That is,

The sum of three or more numbers is the same in whatever manner the numbers are grouped.

This is the **law of grouping**, or the **associative law**, for addition.

59. To add monomials.

EXERCISES

1. Add $4a$ and $3a$.

PROCESS

$$\begin{array}{r} 4a \\ \underline{3a} \\ 7a \end{array}$$

EXPLANATION. — Just as 3 a's and 4 a's are 7 a's, so $3a + 4a = 7a$; that is, when the monomials are similar the sum may be obtained by adding the numerical coefficients and annexing to their sum the common literal part.

Add:

2. $\begin{array}{r} 2x \\ \underline{3x} \end{array}$	3. $\begin{array}{r} a \\ \underline{5a} \end{array}$	4. $\begin{array}{r} -a \\ \underline{4a} \end{array}$	5. $\begin{array}{r} -4c \\ \underline{-3c} \end{array}$
6. $\begin{array}{r} 4v \\ -2v \\ \underline{-7v} \end{array}$	7. $\begin{array}{r} -y \\ 4y \\ \underline{-9y} \end{array}$	8. $\begin{array}{r} 12mb \\ -2mb \\ \underline{-6mb} \end{array}$	9. $\begin{array}{r} 40x^2 \\ -10x^2 \\ \underline{-60x^2} \end{array}$

10. Add $4a$, $\frac{3}{2}a$, $-3a$, and $\frac{1}{2}a$.

PROCESS

$$4a+\tfrac{3}{2}a-3a+\tfrac{1}{2}a=4a-3a+(\tfrac{3}{2}a+\tfrac{1}{2}a)=a+2a=3a.$$

Explanation. — By §§ 57, 58, the numbers may be added in any order or grouped in any manner. For convenience, then, we may first add those with integral coefficients, then those with fractional coefficients, and afterward add these sums, as in the process.

11. Add $5x$, $-\frac{3}{4}x$, $2x$, and $-\frac{1}{4}x$.

12. Add $1\frac{1}{2}ab$, $-2ab$, $5\frac{1}{2}ab$, and $-3ab$.

13. Add $4xyz$, $-\frac{3}{5}xyz$, $\frac{2}{5}xyz$, and $-2\frac{4}{5}xyz$.

14. Add $-c^2de$, $2\frac{1}{6}c^2de$, $\frac{5}{6}c^2de$, and $-5c^2de$.

Simplify:

15. $2y-7y-5y-y+10y-6y+8y$.

16. $5a-3a+8a-10a-5a-11a+24a$.

17. $3by-5by-10by-14by+48by$.

18. $8a^3b+6a^3b-11a^3b-2a^3b+9a^3b$.

19. $1\frac{4}{5}x^2y^2-\frac{1}{2}x^2y^2-1\frac{7}{10}x^2y^2+3\frac{1}{2}x^2y^2+x^2y^2$.

20. $5(xy)^2-3(xy)^2-15(xy)^2+4(xy)^2+13(xy)^2$.

21. $(a-x)+5(a-x)+7(a-x)-3(a-x)-2(a-x)$.

22. $3x(x^2-2x+3)-x(x^2-2x+3)+2x(x^2-2x+3)$.

23. $2(x-1)-13(x-1)+5(x-1)+10(x-1)+6(x-1)$.

Since only similar terms can be united into a single term, dissimilar terms are considered to have been added when they have been written in succession with their proper signs.

24. Add $6a$, $-5b$, $-2a$, $3b$, $2c$, and $-a$.

Solution. — Sum $= 6a-2a-5b+3b+2c-a = 3a-2b+2c$.

Add:

25. $2xy$, $4ab$, $3xy$, and ab.

26. mn, $-3cd$, $-6mn$, and $4cd$.

27. a, $-b$, $2c$, $-2a$, $3b$, and $-4c$.

28. $2a$, $2b$, $2c$, $2d$, $-a$, $-3b$, $-c$, and $-3d$.

60. To add polynomials.

EXERCISES

1. Add $3a-3b+5c$, $-3a+2b$, and $c-4b+2a$.

PROCESS

$$\begin{array}{r} 3a-3b+5c \\ -3a+2b \\ 2a-4b+c \\ \hline 2a-5b+6c \end{array}$$

EXPLANATION. — For convenience, similar terms may be written in the same column (§ 57).

The algebraic sum of the first column is $2a$, of the second $-5b$, of the third $+6c$; and these numbers written in succession express in its simplest form the sum sought.

2. Simplify $11a^2b-7ab^2+2ac^2+10ab-4ac^2+5a^2b-4ab^2+5ac^2+b^3+9ab^2-7a^2b-2b^3+2ab^2-8ab-6a^2b$.

PROCESS

$$\begin{array}{rrrrr} 11a^2b & -7ab^2 & +2ac^2 & +10ab & +b^3 \\ 5a^2b & -4ab^2 & -4ac^2 & & -2b^3 \\ -7a^2b & +9ab^2 & +5ac^2 & & \\ -6a^2b & +2ab^2 & & -8ab & \\ \hline 3a^2b & & +3ac^2 & +2ab & -b^3 \end{array}$$

RULE. — *Arrange the terms so that similar terms shall stand in the same column.*

Find the algebraic sum of each column, and write the results in succession with their proper signs.

3. Add $2a-3b$, $2b-3c$, $5c-4a$, $10a-5b$, and $7b-3c$.

4. Add $x+y+z$, $x-y+z$, $y-z-x$, $z-x-y$, and $x-z$.

Simplify the following polynomials:

5. $7x-11y+4z-7z+11x-4y+7y-11z-4x+y-x-z$.

6. $a+3b+5c-6a+d+4b-2c-2b+5a-d+a-b$.

7. $4x^2-3xy+5y^2+10xy-17y^2-11x^2-5xy+12x^2-2xy$.

8. $2xy-5y^2+x^2y^2-7xy+3y^2-4x^2y^2+5xy+4y^2+x^2y^2$.

9. $2\,ay - 3\,ac - 4\,ay + 4\,ac - 6\,ay + 5\,ac + 11\,ay - 4\,ac - ay.$

10. $5\,am - 3\,a^2m^2 + 4 - 4\,am + a^2m^2 - 2 + 5 + a^2m^2 - 6 + 3\,am + 2\,a^2m^2 - 3\,am - 3.$

11. $6\sqrt{x} - 5\sqrt{xy} + 3\sqrt{y} - 4\sqrt{x} + 6\sqrt{xy} - \sqrt{x} - \sqrt{y} + 3\sqrt{y} - 2\sqrt{xy} + \sqrt{x} + 2\sqrt{xy} - 3\sqrt{y}.$

Add:

12. $7\,a - 3\,b + 5\,c - 10\,d$, $2\,b + d - 3\,c - 4\,e$, $5\,c - 6\,a + 2\,d - 4\,e$, $8\,b - 7\,a - 8\,c - e$, $a - 5c + 5\,d + 11\,e$, $a - b + c + 2\,d + e$, and $5\,a - 4\,b + 2\,c$.

13. $5\,x - 3\,y - 2\,z$, $4\,y - 2\,x + 6\,z$, $3\,a - 2\,x - 4\,y$, $4\,b - 2\,z - 5\,y$, $a - 5\,b$, $5\,y - 6\,x$, $8\,x + 2\,y - 5\,a - 2\,b$, and $6\,x - y - 2\,z + 4\,b$.

14. $2\,c - 7\,d + 6\,n$, $11\,m - 3\,c - 5\,n$, $7\,n - 2\,d - 8\,c$, $8\,d - 3\,m + 10\,c$, $4\,d - 3\,n - 8\,m$, $m - 6\,n$, and $2\,m - 3\,d$.

15. $4\,x^3 - 2\,x^2 - 7\,x + 1$, $x^3 + 3\,x^2 + 5\,x - 6$, $4\,x^2 - 8\,x^3 + 2 - 6\,x$, $2\,x^3 - 2\,x^2 + 8\,x + 4$, and $2\,x^3 - 3\,x^2 - 2\,x + 1$.

16. $a^5 + 5\,a^4b + 5\,ab^4 + b^5$, $a^4b - 2\,a^5 + a^3b^2 - 2\,b^5$, $a^3b^2 - 3\,a^2b^3 - 4\,a^4b - a^5$, and $2\,a^5 + a^4b - 2\,a^3b^2 + 2\,a^2b^3 - 3\,ab^4 + b^5$.

17. $5\,x^6 - x^3 + 7\,x - 9$, $4\,x^5 - 3\,x^6 + 6\,x^2 + 12$, $x^6 - 5\,x^5 - x - 7$, $4 - x^2 - x^6$, $4\,x^4 - 10\,x^2 + 3\,x^6 + 4$, and $x^6 + x^3 - 3\,x^2 - 4\,x - 5$.

18. $3(a+b) + 6(b+c)$, $5(a+b) - 10(b+c)$, $2(a+b) + (b+c)$, $3(b+c) - (a+b)$, $2(b+c) - 10(a+b)$, and $3(a+b) - 3(b+c)$.

19. $x + 3(a+1) - y$, $-(a+1) - 2\,x + 4\,y$, and $3\,x - 4(a+1)$.

20. $a^3 - 3\,a^2bc - 6\,ab^2c$, $a^2b - b^3 - c^3 - 3\,abc$, $ab^2 + b^2c + bc^2$, $5\,a^2bc + 4\,ab^2c + c^3$, $b^3 - a^2b - ab^2$, $a^3 + b^2c + bc^2$, and $2\,ab^2c - 2\,bc^2$.

21. $.12\,x^3 - 4\,x^2 + x + 2$, $.4\,x^2 - 4\,x + .4 - x^3$, $3\frac{1}{2}\,x - .6 + 3\,x^2 + 2\,x^3$, and $1 - \frac{1}{2}\,x + 1.2\,x^2 + \frac{22}{25}\,x^3$.

22. $ax - \frac{3}{2}\,ax^2 - \frac{1}{4}\,ax^3$, $\frac{3}{2}\,ax^3 - \frac{1}{2}\,ax^2 - \frac{1}{6}\,bxy$, $\frac{1}{2}\,bxy - \frac{3}{4}\,ax^3 - \frac{7}{8}\,ab$, $\frac{2}{3}\,bxy - \frac{5}{6}\,ab + \frac{7}{2}\,ax$, and $2\,ab - \frac{3}{2}\,ax + \frac{5}{4}\,ax^2$.

SUBTRACTION

EXERCISES

61. **1.** From $10x$ subtract $15x$.

PROCESS

$$\begin{array}{r} 10x \\ 15x \\ \hline -5x \end{array}$$

EXPLANATION. — Since (§ 56, Prin.) subtracting any number is equivalent to adding it with its sign changed, $15x$ may be subtracted from $10x$ by changing the sign of $15x$ and adding $-15x$ to $10x$.

	2.	**3.**	**4.**	**5.**	**6.**
From	$12a$	$9am$	$8x^2y^2$	$24mn^2$	$11(a+b)$
Take	$5a$	$21am$	$18x^2y^2$	$12mn^2$	$21(a+b)$

	7.	**8.**	**9.**	**10.**
From	$9a+7b$	$5a+10b$	$10x+2y$	$3m+3n$
Take	$2a+3b$	$7a+4b$	$6x+4y$	$2m+5n$

	11.	**12.**	**13.**	**14.**
From	$15m+n$	$7x+2y$	$4x+4y$	$8p+3q$
Take	$12m+2n$	$4x+4y$	$7x+2y$	$10p+2q$

15. From $8p+3z$ subtract $10p+z$.

16. From $15m+n$ subtract $5m+3n$.

17. From $3ax+5by$ subtract $4ax+6by$.

18. From $8abc+19mx$ subtract $20abc+7mx$.

19. From $a+3b+c$ subtract $a+b+3c$.

20. From $8x - 3y$ subtract $5x - 7y$.

PROCESS

$$\begin{array}{r} 8x - 3y \\ 5x - 7y \\ - \quad + \\ \hline 3x + 4y \end{array}$$

EXPLANATION. — Since (§ 56, Prin.) subtracting any number is equivalent to adding it with its sign changed, subtracting $5x$ from $8x$ is equivalent to adding $-5x$ to $8x$, and subtracting $-7y$ from $-3y$ is equivalent to adding $+7y$ to $-3y$.

RULE. — *Change the sign of each term of the subtrahend, and add the result to the minuend.*

After a little practice the student should make the change of sign mentally.

	21.	22.	23.	24.	25.
From	$5a$	$6xy$	$-9mn$	$-13\sqrt{x}$	$-3(a+b)$
Take	$-2a$	$-3xy$	$-4mn$	$-5\sqrt{x}$	$-10(a+b)$

	26.	27.	28.
From	$4m - 3n + 2p$	$8a - 10b + c$	$3x + 2y - z$
Take	$2m - 5n - p$	$6a - 5b - c$	$5x - 4y - z$

	29.	30.	31.
From	$a - b + c$	$8a^2b - 5ac^2 + 9a^2c$	$r - s + t$
Take	$2a + b - c$	$3a^2b + 2ac^2 - 9a^2c$	$r + s - t$

32. From $5x - 3y + z$ take $2x - y + 8z$.

33. From $3a^2b + b^3 - a^3$ take $4a^2b - 8a^3 + 2b^3$.

34. From $13a^2 + 5b^2 - 4c^2$ take $8a^2 + 9b^2 + 10c^2$.

35. From $15x - 3y + 2z$ subtract $3x + 8y - 9z$.

36. From $a^2 - ab - b^2$ subtract $ab - 2a^2 - 2b^2$.

37. From $m^2 - mn + n^2$ subtract $2m^2 - 3mn + 2n^2$.

38. From $5x^2 - 2xy - y^2$ subtract $2x^2 + 2xy - 3y^2$.

39. From $2ax - by - 5xy$ subtract $2by - 2ax - 3xy$.

40. *From* $4ab + c$ subtract $a^2 - b^2 + abc + 2ab - 2c$.

41. From $2a+c$ subtract $a-b+c$.

42. From $2m+n$ subtract $n-2p$.

43. From $x+y$ subtract $3a-4+y$.

44. From $2x^2+2xy$ subtract x^2-xy-y^2.

45. From $2a-2d$ subtract $a-b+c-d$.

46. From $2b$ subtract $b-a-c-d$.

47. From a^3+x^3 subtract $a^3-3a^2x+3ax^2-x^3$.

48. From a^4+1 subtract $1-a+a^2-a^3+a^4$.

49. From the sum of $3a^2-2ab-b^2$ and $3ab-2a^2$ subtract a^2-ab-b^2.

50. From $3x-y+z$ subtract the sum of $x-4y+z$ and $2x+3y-2z$.

51. From $a+b+c$ subtract the sum of $a-b-c$, $b-c-a$, and $c-a-b$.

52. Subtract the sum of m^2n-2mn^2 and $2m^2n-m^3-n^3+2mn^2$ from m^3-n^3.

53. Subtract the sum of $2c-9a-3b$ and $3b-5a-5c$ from $b-3c+a$.

54. From $3bx+4ay$ subtract the sum of $3ay-4bx$ and $bx+ay$.

55. From the sum of $1+x$ and $1-x^2$ subtract $1-x+x^2-x^3$.

56. From $\frac{2}{3}x^3-\frac{2}{5}x^2+3x-7$ subtract $\frac{1}{2}x^3-\frac{4}{5}x^2+\frac{5}{2}x-10$.

57. From $\frac{1}{8}m^3-\frac{1}{4}m^2n+\frac{1}{3}mn^2-\frac{8}{27}n^3$ subtract $n^3-m^3+\frac{1}{3}mn^2-\frac{1}{2}m^2n$.

58. From $5(a+b)-3(x+y)+4(m+n)$ subtract $4(a+b)+2(x+y)+(m+n)$.

59. From the sum of $3x^2-2x+1$ and $2x-5$ subtract the sum of $x-x^2+1$ and $2x^2-4x+3$.

PARENTHESES

62. Removal of parentheses preceded by + or −.

EXERCISES

1. Remove parentheses and simplify $3a+(b+c-2a)$.

SOLUTION. — The given expression indicates that $(b+c-2a)$ is to be *added* to $3a$. This may be done by writing the terms of $(b+c-2a)$ after $3a$ in succession, each with its proper sign, and uniting terms.

$$\therefore 3a+(b+c-2a)=3a+b+c-2a=a+b+c.$$

2. Remove parentheses and simplify $4a-(2a-2b)$.

SOLUTION. — The given expression indicates that $(+2a-2b)$ is to be *subtracted* from $4a$. Proceeding as in subtraction, that is, changing the sign of each term of the subtrahend and adding, we have

$$4a-(2a-2b)=4a-2a+2b=2a+2b.$$

PRINCIPLES. — 1. *A parenthesis preceded by a plus sign may be removed from an expression without changing the signs of the terms in parenthesis.*

2. *A parenthesis preceded by a minus sign may be removed from an expression, if the signs of all the terms in parenthesis are changed.*

Simplify each of the following:

3. $a+(b-c)$.
4. $a-(b-c)$.
5. $x-(y-z)$.
6. $x-(-y+z)$.
7. $m-n-(-a)$.
8. $m-(n-2a)$.
9. $5x-(2x+y)$.
10. $a-b-(c-d)$.
11. $a-b-(-c+a)$.
12. $a-m-(n-m)$.
13. $5a-2b-(a-2b)$.
14. $a-(b-c+a)-(c-b)$.
15. $2xy+3y^2-(x^2+xy-y^2)$.
16. $m+(3m-n)-(2n-m)+n$.

When an expression contains parentheses within parentheses, they may be removed *in succession*, beginning with either the outermost or the innermost, preferably the latter.

17. Simplify $6x-[3a-\{4b+(8b-2a)-3b\}+4x]$.

SOLUTION

	$6x-[3a-\{4b+(8b-2a)-3b\}+4x]$
Prin. 1,	$=6x-[3a-\{4b+8b-2a-3b\}+4x]$
Uniting terms,	$=6x-[3a-\{9b-2a\}+4x]$
Prin. 2,	$=6x-[3a-9b+2a+4x]$
Uniting terms,	$=6x-[5a-9b+4x]$
Prin. 2,	$=6x-5a+9b-4x$
Uniting terms,	$=2x-5a+9b$.

Simplify each of the following:

18. $4a+b-\{x+4a+b-2y-(x+y)\}$.

19. $ab-\{ab+ac-a-(2a-ac)+(2a-2ac)\}$.

20. $a+[y-\{5+4a-(6y+3)\}-(7y-4a-1)]$.

21. $4m-[p+3n-(m+n)+3-(6p-3n-5m)]$.

22. $a+2b+(14a-5b)-\{6a+6b-(5a-\overline{4a-4b})\}$.

23. $12a-\{[4-3b-(6b+3c)]+b-8-(5a-2b-6)\}$.

24. $a+b-\{-[a+b-(c+x)]-[3a-(c-x+a)-b]+4a\}$.

25. $x^3-[x^2-(1-x)]-\{1+[x^2-(1-x)+x^3]\}$.

26. $4-\{[5y-(3-\overline{2x-2})]-[x+(5y-\overline{x+3})]\}$.

27. $ab-\{5+x-(b+c-ab+x)\}+[x-(b-c-7)]$.

28. $-\{3ax-[5xy-3z]+z-(4xy+[6z+7ax]+3z)\}$.

29. $1-x-\{1-x-[1-x-(1-x)-(x-1)]-x+1\}$.

30. $1-x-\{1-[x-1+(x-1)-(1-x)-x]+1-x\}$.

31. $a-(b-c)-[a-\{b-c-(b+c-a)+(a-b)+(c-a)\}]$.

63. Grouping terms by means of parentheses.

It follows from § 62 that:

PRINCIPLES. — 1. *Any number of terms of an expression may be inclosed in a parenthesis preceded by a plus sign without changing the signs of the terms to be inclosed.*

2. *Any number of terms of an expression may be inclosed in a parenthesis preceded by a minus sign, if the signs of the terms to be inclosed are changed.*

EXERCISES

64. 1. In $a^2 + 2ab + b^2$, group the terms involving b.

SOLUTION

$$a^2 + 2ab + b^2 = a^2 + (2ab + b^2).$$

2. In $a^2 - x^2 - 2xy - y^2$, group as a subtrahend the terms involving x and y.

SOLUTION

$$a^2 - x^2 - 2xy - y^2 = a^2 - (x^2 + 2xy + y^2).$$

3. In $ax^2 + ab + 2x^2 + 2b$, group the terms involving x^2, and also the terms involving b, as addends.

4. In $a^3 + 3a^2b + 3ab^2 + b^3$, group the first and fourth terms, and also the second and third terms, as addends.

In each of the following expressions group the last three terms as a subtrahend:

5. $a^2 - b^2 - 2bc - c^2$.

6. $a^2 - b^2 + 2bc - c^2$.

7. $a^2 + 2ab + b^2 - c^2 + 2cd - d^2$.

8. $a^2 - 2ab + b^2 - c^2 - 2cd - d^2$.

9. In $a^2 + 2ab + b^2 - 4a - 4b + 4$, group the first three terms as an addend and the fourth and fifth as a subtrahend.

Errors like the following are common. Point them out.

10. $x^3 - x^2 + x - 1 = (x^3 - 1) - (x^2 + x)$.

11. $x^2 - y^2 + 2yz - z^2 = x^2 - (y^2 + 2yz - z^2)$.

65. Collecting literal coefficients.

EXERCISES

Add:

1. $\begin{array}{r} ax \\ bx \\ \hline (a+b)x \end{array}$

2. $\begin{array}{r} bm \\ -cm \\ \hline (b-c)m \end{array}$

3. $\begin{array}{r} -cx \\ -dx \\ \hline -(c+d)x \end{array}$

4. $\begin{array}{l} ax \\ \underline{nx} \end{array}$

5. $\begin{array}{r} cy \\ \underline{-dy} \end{array}$

6. $\begin{array}{l} -mp \\ \underline{-np} \end{array}$

Subtract the lower expression from the upper one:

7. $\begin{array}{l} mx \\ \underline{nx} \end{array}$

8. $\begin{array}{l} dy+az \\ \underline{ey-bz} \end{array}$

9. $\begin{array}{r} ax-by \\ \underline{2x-cy} \end{array}$

10. $\begin{array}{l} a^2x+aby \\ \underline{b^2x+aby} \end{array}$

11. $\begin{array}{r} mx-ny \\ \underline{nx-my} \end{array}$

12. $\begin{array}{r} ax-5y \\ \underline{5x+ay} \end{array}$

13. Collect the coefficients of x and of y in $ax - ay - bx - by$.

SOLUTION. — The total coefficient of x is $(a-b)$; the total coefficient of y is $(-a-b)$, or $-(a+b)$.

$$\therefore ax - ay - bx - by = (a-b)x - (a+b)y.$$

Collect in alphabetical order the coefficients of x and of y in each of the following, giving each parenthesis the sign of the first coefficient to be inclosed therein:

14. $ax - by - bx - cy + dx - ey$.
15. $5ax + 3ay - 2dx + ny - 5x - y$.
16. $cx - 2bx + 7ay + 3ax - lx - ty$.
17. $bx + cy - 2ax + by - cx - dy$.
18. $bx - cy - 2ay + by$.
19. $rx - ay - sx + 2cy$.
20. $x^2 + ax - y^2 + ay$.
21. $x^2 - ay - ax - y^2$.

Group the same powers of x in each of the following:

22. $ax^3 + bx^2 - cx + ex^3 - dx^2 - fx$.
23. $x^3 + 3x^2 + 3x - ax^2 - 3ax^3 + bx$.
24. $x^2 - abx - x^3 - bx^2 - cx - mnx^3 + dx$.
25. $ax^4 - x^4 - ax^2 + x^2 + ax - x - abx^3 + x^3$.

TRANSPOSITION IN EQUATIONS

66. In an equation, the number on the left of the sign of equality is called the **first member** of the equation, and the number on the right is called the **second member**.

In the equation $x = 6 + 2$, x is the first member and $6 + 2$ is the second.

67. Observe how (2) is obtained from (1) in each of the following:

1.	$-2+5=3$ (1)	3.	$6=6$ (1)
Adding	$2 \quad =2$	Multiplying by	$2=2$
Sums,	$5=5$ (2)	Products,	$12=12$ (2)
2.	$4+3=7$ (1)	4.	$8=8$ (1)
Subtracting	$4 \quad =4$	Dividing by	$4=4$
Remainders,	$3=3$ (2)	Quotients,	$2=2$ (2)

The following principles, illustrated above, are useful in solving equations. They are so simple as to be self-evident. Such self-evident principles are called **axioms**.

68. Axioms.—1. *If equals are added to equals, the sums are equal.*

2. *If equals are subtracted from equals, the remainders are equal.*

3. *If equals are multiplied by equals, the products are equal.*

4. *If equals are divided by equals, the quotients are equal.*

In the application of Ax. 4, it is not allowable to divide by zero (§ 547).

EXERCISES

69. 1. Solve $x-6=4$ by adding 6 to both members (Ax. 1).

2. Solve the equation $x+3=11$ by employing Ax. 2.

3. Solve $\frac{1}{3}x=10$ by employing Ax. 3.

4. Solve $7x=21$. Explain how Ax. 4 applies.

5. Solve $\frac{2}{3}x=16$ in two steps, first finding the value of $\frac{1}{3}x$ by Ax. 4, then the value of x by Ax. 3.

Solve, and give the axiom applying to each step:

6. $2x=6.$	17. $x+2=10.$	28. $\frac{3}{2}m=9.$
7. $5x=5.$	18. $w-5=11.$	29. $\frac{4}{3}n=8.$
8. $4y=8.$	19. $w+1=12.$	30. $\frac{5}{2}x=10.$
9. $3y=9.$	20. $s-7=10.$	31. $\frac{3}{5}x=21.$
10. $\frac{1}{2}z=5.$	21. $9+s=12.$	32. $\frac{5}{6}z=20.$
11. $\frac{1}{3}z=2.$	22. $5+y=15.$	33. $\frac{5}{8}z=15.$
12. $\frac{1}{4}v=3.$	23. $10+x=12.$	34. $5m-1=9.$
13. $8v=24.$	24. $11+x=15.$	35. $4h+3=7.$
14. $9r=54.$	25. $20+x=30.$	36. $6r-7=5.$
15. $\frac{1}{4}r=1.5.$	26. $7y-5=2.$	37. $\frac{1}{2}b+3=8.$
16. $\frac{1}{2}x=2.5.$	27. $2z+3=9.$	38. $\frac{1}{5}x+2=6.$

70. 1. Adding 7 to both members of the equation

$$x-7=3,$$

we obtain, by Ax. 1, $\quad x=3+7.$

-7 has been removed from the first member, but reappears in the second member with the opposite sign.

2. Subtracting 5 from both members of the equation

$$x+5=9,$$

we obtain, by Ax. 2, $\quad x=9-5.$

When plus 5 is removed, or **transposed**, from the first member to the second, its sign is changed.

3. Explain the transposition of terms in each of the following:

$2x-1=5;$	$3x+2=11;$	$4x=14-3x;$
$2x=5+1.$	$3x=11-2.$	$4x+3x=14.$

71. PRINCIPLE. — *Any term may be transposed from one member of an equation to the other, provided its sign is changed.*

EXERCISES

72. **1.** Solve the equation $2x + 20 = 80 - 4x$.

PROCESS

$$2x + 20 = 80 - 4x$$
$$2x + 4x = 80 - 20$$
$$6x = 60$$
$$x = 10$$

EXPLANATION. — The first step in solving an equation is to collect the unknown terms into one member (usually the first member) and the known terms into the other member.

Adding $4x$ to both members, or transposing $-4x$ from the second member to the first and changing its sign, places all unknown terms in the first member.

Subtracting 20 from both members, or transposing $+20$ from the first member to the second and changing its sign, places all known terms in the second member.

Uniting similar terms and dividing both members by 6, the coefficient of x, we find the value of x to be 10.

VERIFICATION. — The result should always be verified by substituting it for the unknown number *in the given equation.* If the members of the given equation reduce to the same number, the result obtained is correct.

Substituting 10 for x, makes the first member $20 + 20$, or 40, and the second member $80 - 40$, or 40. Hence, 10 is the true value of x.

2. Solve the equation $7 - 5x = 7 - 30$.

FIRST PROCESS

$$\cancel{7} - 5x = \cancel{7} - 30$$
$$-5x = -30$$
$$5x = 30$$
$$x = 6$$

SECOND PROCESS

$$\cancel{7} - 5x = \cancel{7} - 30$$
$$30 = 5x$$
$$6 = x \text{ or } x = 6$$

TEST. $7 - 5 \cdot 6 = 7 - 30$

SUGGESTIONS. — 1. By Ax. 2 the same number may be subtracted, or *canceled*, from both members.

2. By Ax. 2 the signs of all the terms of an equation may be changed (first process); for each member may be subtracted from the corresponding member of the equation $0 = 0$.

3. To avoid multiplying by -1, the second process may be adopted.

Solve and verify:

3. $3=5-x.$
4. $9-5x=-1.$
5. $10+v=18-v.$
6. $2z+2=32-z.$
7. $7x+2=x+14.$
8. $3p+2=p+30.$
9. $5m-5=2m+4.$
10. $6y-2=3y+7.$
11. $8x-7=3+6x.$
12. $8+7a=5a+20.$
13. $2+13h=50-9.$
14. $50-n=20+n.$
15. $3x-23=x-17.$
16. $4x+12=2x+36.$
17. $2x+\frac{1}{2}x=30-\frac{1}{2}x.$
18. $3x-\frac{1}{4}x=30+\frac{1}{4}x.$
19. $5w-10=\frac{2}{3}w+16.$
20. $4r-18=20+\frac{1}{5}r.$

Simplify as much as possible before transposing terms, solve, and verify:

21. $10x+30-4x-(9x-33-12x)=90+12-4x.$
22. $16x+12-75+2x-12-70=8x-(50+25).$
23. $11s-60+5s+17-(2s-41)-3s=2s+97.$
24. $10z-35-(12z-16)+32=4z-(35-10z)+32.$
25. $36+5x-22-(7x-16)=5x+17-(2x+22).$
26. $12x-(6x-17x-15-x)=15-(2-17x+6x-4-12x).$
27. $14n-35=9-(11n-4-16+10n-n)+136-16n.$

Algebraic Representation

73. 1. Express the sum of x, $-y$, and $-z$.

2. What number is n less than 25?

3. Express the number that exceeds a by b.

4. How much does z exceed $10+y$?

5. What number must be added to m so that the sum shall be 4?

6. Mary read 10 pages in a book, stopping at the top of page a. On what page did she begin to read?

7. A man made three purchases of a, b, and 2 dollars, respectively, and tendered a 10-dollar bill. Express the number of dollars in change due him.

8. A has x dollars and B, y dollars. If A gives B m dollars, how much will each then have?

9. If 40 is separated into two parts, one of which is x, represent the other part.

10. What two whole numbers are nearest to x, if x is a whole number? to $a+b$, if $a+b$ is a whole number?

11. If x is an even number, what are the two even numbers nearest to x? the two odd numbers nearest to x?

12. If $n+2$ is an odd number, what are the two odd numbers nearest to $n+2$? the two even numbers?

13. There is a family of three children, each of whom is 2 years older than the next younger. When the youngest is x years old, what are the ages of the others? When the oldest is y years old, what are the ages of the others?

14. A boy who had x dollars spent y cents of his money. How much money had he left?

15. The number 25 may be written $20+5$. Write the number whose first digit is x and second digit y.

16. The number 376 may be written $300+70+6$. Write the number whose first digit is x, second digit y, and third digit z.

SOLUTION OF PROBLEMS

74. If $3x =$ a certain number and $x+10 =$ the same number, then,

$$3x = x + 10.$$

This illustrates another axiom to be added to the list in § 68. It will be found useful in the solution of problems.

Axiom 5. — *Numbers that are equal to the same number, or to equal numbers, are equal to each other.*

75. Illustrative Problem. — Of the steam vessels built on the Great Lakes one year, 21, or 5 less than $\frac{1}{3}$ of all, were of steel. How many steam vessels were built on the Lakes that year?

SOLUTION. — Let	$x =$ the number of steam vessels built.
Then,	$\frac{1}{3}x - 5 =$ the number of steel vessels.
But	$21 =$ the number of steel vessels.
$\therefore$ Ax. 5,	$\frac{1}{3}x - 5 = 21.$
Transposing,	$\frac{1}{3}x = 21 + 5 = 26.$
Hence, Ax. 3,	$x = 78$, the number of steam vessels built.

76. A problem gives certain conditions, or relations, that exist between known numbers and one or more unknown numbers. The statement in algebraic language of these conditions is called the **equation** of the problem.

The equation of the problem just solved is $\frac{1}{3}x - 5 = 21$.

77. General Directions for Solving Problems. — 1. *Represent one of the unknown numbers by some letter, as x.*

2. *From the conditions of the problem find an expression for each of the other unknown numbers.*

3. *Find from the conditions two expressions that are equal and write the equation of the problem.*

4. *Solve the equation.*

Problems

78. 1. What number increased by 10 is equal to 19?

2. What number diminished by 30 is equal to 20?

3. What number diminished by 111 is equal to -15?

4. What number exceeds $\frac{1}{3}$ of itself by 10?

5. Five times a number exceeds 3 times the number by 14. What is the number?

6. If 5 is subtracted from a certain number, and the difference is subtracted from 3 times the number, the result is 35. What is the number?

7. The double of a number is 64 less than 10 times the number. What is the number?

8. The sum of a number and .04 of itself is 46.8. What is the number?

9. What number decreased by .35 of itself equals 52?

10. If 4 is subtracted from a certain number, and the difference is subtracted from 40, the result is 3 times the number. What is the number?

11. Three times a certain number is as much less than 72 as 4 times the number exceeds 12. What is the number?

12. Twice a certain number exceeds $\frac{1}{8}$ of the number as much as 6 times the number exceeds 65. What is the number?

13. A prime dark sea-otter skin cost $400 more than a brown one. If the first cost 3 times as much as the second, how much did each cost?

14. The total height of a certain brick chimney in St. Louis is 172 feet. Its height above ground is 2 feet more than 16 times its depth below. How high is the part above ground?

15. A wagon loaded with coal weighed 4200 pounds. The coal weighed 1800 pounds more than the wagon. How much did the wagon weigh? the coal?

16. It cost a mine owner $1.90 per ton to mine soft coal and ship it to market. The cost of shipping the coal was $.10 more per ton than the cost of mining it. Find the cost of mining it.

17. A mining company sold copper ore at $5.28 per ton. The profit per ton was $.22 less than the expense. What was the profit on each ton?

18. Recently Germany had 1025 ships of over 1000 tons capacity. There were 25 more than $\frac{1}{4}$ as many sailing vessels as steamers. Find the number of each.

19. The length of the steamship *Lusitania* is 790 feet, or 2 *feet less than 9 times* its width. What is the width, or beam, *of the vessel?*

20. The Canadian, or Horseshoe, Falls, in the Niagara River, are 158 feet high. This is 8 feet more than $\frac{10}{11}$ of the height of the American Falls. Find the height of the American Falls.

21. The width of the St. Lawrence River at Quebec at a point where it is spanned by a bridge is 1800 feet. This is 180 feet less than $\frac{5}{6}$ of the length of the bridge. How long is the bridge?

22. An American cargo consisting of 24,470 barrels and boxes of apples was landed at Bremen, Germany. There were 170 less than 15 times as many barrels as boxes. Find the number of barrels and the number of boxes.

23. In lighting a hall a certain number of 16-candle power electric lamps and twice as many 20-candle power lamps were used. The total illumination amounted to 224 candle power. Find the number of lamps of each kind used.

24. The principal sources of the world's cotton crop one year were the United States and India, which together supplied 16,700,000 bales. The former supplied 50,000 bales more than $3\frac{1}{2}$ times as much as the latter. How much cotton did each country produce that year?

25. At the waterworks 2 large pumps and 4 small ones delivered 4800 gallons of water per minute. Each of the large pumps delivered 4 times as much water as each small pump. How many gallons per minute did each small pump deliver? each large pump?

26. The cost of construction for 4 miles of macadam road was $2400 more than for 8 miles of gravel road. The latter cost $\frac{2}{5}$ as much per mile as the former. What was the cost of each per mile?

27. The total duty on a quantity of figs, raisins, and dates, imported into the United States, was $50. The duty on the raisins was $1\frac{1}{4}$ times as much as on the figs, and that on the dates $\frac{1}{4}$ as much as on the figs. What was the duty on the figs? the raisins? the dates?

REVIEW

79. 1. Define square of a number; square root of a number. Show the difference between these by an illustrative example.

2. Distinguish between power and exponent; between exponent and coefficient.

3. By what law do we know that the sum of $2x, -3y, 4x$, and $5y$ is the same as the sum of $2x, 4x, 5y$, and $-3y$?

4. From the sum of $z + xy$ and $z - x^2y$ subtract the sum of $xy - z$, $x^3 - x^2y$, and $y^3 - x^3$.

5. From the sum of $3x^2 - 2xy + y^2$ and $2xy - 5y$ subtract $2x^2 - 6xy + 4y^2$ less $x^2 - xy + y^2$.

6. What number added to $-a^2 + b^2 - 2ab$ gives 0 for the sum?

7. Give the general name that belongs to the two following expressions, and the specific name of each:

$$x^2 - xy + y^2 \text{ and } x^3 - x^2y.$$

8. Remove parentheses and then regroup the terms in order, first as binomials; second as trimonials:

$$2a - [2x^3 + 2b - c - \{-2d - (2y^2 - z) + m\}] + n^3.$$

9. Inclose the last four terms of $a - y - b + x - 2$ in brackets and the last two terms in parentheses.

10. Collect similar terms within parentheses:

$$ax^2 - cy + ax - 2ax^2 + 2cy^2 - ax - cy^2 + ax^2 + cy.$$

11. How does $kl \div yz$ differ in meaning from $k \times l \div y \times z$?

12. Define positive numbers; negative numbers. Illustrate.

13. The temperature at White River Junction one winter day was $-40°$ F. The temperature at Washington that day was $40°$ F. What was the difference in temperature?

What is the difference in the absolute value of the numbers *denoting these two* temperatures? in their relative value?

When $a=1, b=2,\ c=3,\ d=4$, and $e=5$, find the value of:

14. $a-(e+b)-(c+d)-(e-d+b+c)$.

15. $3\,ab^2-2\,bc^3-(d^ce^a-ac^2)+8\,be^2$.

16. $5\,ac+b\sqrt{d}\div a+2(a-b)(d-e)+bce$.

17. $\sqrt{2\,edb+4\,e}-\sqrt[3]{9\,a^cc^4}-2\,c^2d-(abe-abcde)$.

18. Show that a number may be transposed from one member of an equation to the other, if its sign is changed.

Solve, giving reasons for each step:

19. $5\,x+5=7\,x-3$.

20. $2\,x-(4+x)-5\,x+20=4\,x+(4-5\,x)$.

21. $3\,x-5-6\,x+1-(9\,x-5-5\,x)=3\,x-9$.

22. $10\,x-3-(4-2\,x)+(3\,x-4\,x+5-2\,x)=2-3\,x+4\,x-(2\,x+x)+7$.

23. Prove that $5\,x+4=6\,x-1$, when $x=5$.

24. Prove that $17-x=x-19$, when $x=18$.

25. Show that $x(x-y+xy)=x^2-xy+x^2y$, for as many numerical values as you may substitute for x and y.

Supply the missing coefficients in the following equations:

26. $3\,a-{*}b+6\,a+5\,b-{*}xy={*}a+b-2\,xy$.

27. $x^2+2\,xy+3\,y^2-[2\,x^2+{*}\,y^2]={*}\,x^2+{*}\,xy$.

28. $6\,m^2+9\,mn-3\,n^2-[3\,m^2+{*}\,mn]+{*}\,n^2={*}\,m^2-mn-2\,n^2$.

29. From $a^{2m}b^n+a^mb^{2n}-a^{3n}$ take $-2\,a^{2m}b^n+a^{3n}$.

30. If $x=r^2+rs-s^2$, $y=2\,r^2+4\,rs+2\,s^2$, and $z=r^2-3\,rs-s^2$, find the value of $x+y-z$.

31. Add $.5\,m^3+2.5\,m^2n+n+3$, $-.6\,mn^2+.5\,m^3+.5\,n-3$, and $-m^3-2.5\,m^2n-1.5\,n+1$.

32. Add $.5\,x^4-3\frac{1}{2}\,y^3+\frac{3}{4}\,z^4$, $-3\frac{1}{4}\,z^4-4\,x^4+.5\,y^3$, $-\frac{3}{4}\,y^3-z^4+x^4$, and $9\frac{1}{2}\,x^4-1\frac{1}{2}\,z^4+2.25\,y^3$.

MULTIPLICATION

80. As in arithmetic, the number multiplied is called the **multiplicand**; the number by which the multiplicand is multiplied, the **multiplier**; and the result, the **product.**

When the multiplier is a positive integer, **multiplication** may be defined as the process of taking the multiplicand additively as many times as there are ones in the multiplier.

Thus, since $3 = 1 + 1 + 1$, $5 \cdot 3 = 5 + 5 + 5 = 15$.

Since fractional and negative multipliers cannot be obtained by simple repetitions of 1, the definition of multiplication must now be stated in more general terms.

Although fractional and negative multipliers cannot be obtained directly from the *primary* unit 1, they may be obtained from units *derived* from 1, by taking a part of 1, or by changing the sign of 1, or by both operations.

Thus, $\frac{3}{4} = +\frac{1}{4} + \frac{1}{4} + \frac{1}{4}$ and $-\frac{3}{4} = (-\frac{1}{4}) + (-\frac{1}{4}) + (-\frac{1}{4})$.

In multiplying 5 by 3, we first observe how 3 is derived from the primary unit 1; then in this process we replace 1 by 5.

Therefore, in multiplying 5 by $\frac{3}{4}$, in order to extend the definition of multiplication in harmony with the existing one, having observed that the multiplier $\frac{3}{4}$ is derived from the primary unit by taking 3 of the 4 equal parts of it, we should take 3 of the 4 equal parts of the multiplicand 5.

Thus, since $\frac{3}{4} = \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$, $5 \cdot \frac{3}{4} = \frac{5}{4} + \frac{5}{4} + \frac{5}{4} = \frac{15}{4}$. Similarly, $-3 = (-1) + (-1) + (-1)$; $\therefore 5 \cdot (-3) = (-5) + (-5) + (-5) = -15$.

Multiplication is the process of finding a number that is obtained from the multiplicand just as the multiplier is obtained from unity.

81. Arithmetically, $2 \times 3 = 3 \times 2$.

In general, $ab = ba$. That is,

The factors of a product may be taken in any order.

This is the **law of order**, or the **commutative law**, for multiplication.

82. $2 \times 3 \times 5 = (2 \times 3) \times 5 = 2 \times (3 \times 5) = (2 \times 5) \times 3$.

In general, $abc = (ab)c = a(bc) = (ac)b$. That is,

The factors of a product may be grouped in any manner.

This is the **law of grouping**, or the **associative law**, for multiplication.

83. **Sign of the product.**

(1) Suppose that the *multiplier* is a *positive* number, as $+2$.

Since $+2$ may be obtained from $+1$ by taking $+1$ additively 2 times, a process that involves *no change of sign*, by the definition of multiplication any number may be multiplied by $+2$ by taking the number, *with its own sign*, additively 2 times.

$$+4 \cdot 2 = (+4) + (+4) = +8; \qquad -4 \cdot 2 = (-4) + (-4) = -8.$$

The product, therefore, has the *same sign* as the multiplicand.

(2) Suppose that the *multiplier* is a *negative* number, as -2.

Since -2 may be obtained from $+1$ by *changing the sign* of $+1$ and taking the result additively 2 times, any number may be multiplied by -2 by *changing the sign* of the number and taking the result additively 2 times.

$$+4 \cdot -2 = (-4) + (-4) = -8; \qquad -4 \cdot -2 = (+4) + (+4) = +8.$$

The product has the *sign opposite* to that of the multiplicand.

(3) The conclusions of (1) and (2) may be written as follows:

From (1), $+a$ multiplied by $+b = +ab$,

and $-a$ multiplied by $+b = -ab$;

from (2), $+a$ multiplied by $-b = -ab$,

and $-a$ multiplied by $-b = +ab$. Whence,

84. **Law of Signs for Multiplication.** — *The sign of the product of two factors is + when the factors have* **like signs**, *and — when they have* **unlike signs**.

EXERCISES

85. 1. Multiply each of the following by $+2$; then by -2:

$3,\ 5,\ -6,\ 10,\ -8,\ -9,\ -12,\ a,\ x,\ -b.$

2.
Multiply	-8	9	6	4	-2
By	6	3	-5	-7	10

3.
Multiply	a	$-b$	$-x$	$-y$	n
By	4	6	-8	-1	-12

86. Product of two monomials.

The product of two numbers must contain all the factors, numerical and literal, of both numbers. These may be taken in any order or associated in any manner (§§ 81, 82).

Usually the numerical coefficients are grouped, to form the coefficient of the product; then the literal factors are written, any like factors that may exist being grouped by exponents.

Thus, § 27, $4\,m^2n \cdot 3\,m^3n^2 = 4 \cdot m \cdot m \cdot n \cdot 3 \cdot m \cdot m \cdot m \cdot n \cdot n$

§ 82, $= (4 \cdot 3)(m \cdot m \cdot m \cdot m \cdot m)(n \cdot n \cdot n)$

§ 27, $= 12\,m^5n^3.$

87. Law of Coefficients for Multiplication. — *The coefficient of the product is equal to the product of the coefficients of multiplicand and multiplier.*

88. Law of Exponents, or Index Law, for Multiplication. — *The exponent of a number in the product is equal to the sum of its exponents in multiplicand and multiplier.*

The *proof* for *positive integral exponents* follows:

Let m and n be any positive integers, and let a be any number.

By notation, § 27,

$$a^m = a \cdot a \cdot a \cdots \text{to } m \text{ factors},$$

and $$a^n = a \cdot a \cdot a \cdots \text{to } n \text{ factors};$$

$$\therefore a^m \cdot a^n = (a \cdot a \cdot a \cdots \text{to } m \text{ factors})(a \cdot a \cdot a \cdots \text{to } n \text{ factors})$$

by assoc. law, $= a \cdot a \cdot a \cdots \text{to } (m+n) \text{ factors}$

by notation, $= a^{m+n}.$

EXERCISES

89. **1.** Multiply $-4\,ax^2$ by $2\,a^3x^4$.

PROCESS

$$\begin{array}{r} -4\,ax^2 \\ 2\,a^3x^4 \\ \hline -8\,a^4x^6 \end{array}$$

EXPLANATION. — Since the signs of the monomials are unlike, the sign of the product is minus (Law of Signs).

$4 \cdot 2 = 8$ (Law of Coefficients).

$a \cdot a^3 = a^1 \cdot a^3 = a^{1+3} = a^4$ (Law of Exponents).

$x^2 \cdot x^4 = x^{2+4} = x^6$ (Law of Exponents).

Hence, the product is $-8\,a^4x^6$.

Multiply:

2. $10\,a^5$ $5\,a^3$	3. $-5\,m^3n^2$ $3\,mn$	4. $-4\,abc$ $2\,a^2b$	5. $3\,a^2bc^3$ $-7\,ab^2c$
6. x^2y^2 xy^3	7. $5\,pq^2x^2$ $-2\,rq^4x$	8. $-8\,ab$ -1	9. $-5\,a^2x^2$ $-2\,ax^3$
10. $-2\,x$ $2\,x^2$	11. $-6\,a^2c^2x$ $-4\,a^3bn$	12. $-3\,ab$ $2\,ba$	13. $-2\,a^6x^2$ $-4\,ax^4$
14. $-3\,n^3$ $6\,b^3$	15. $4\,a^2b^3y^4$ $3\,a^3b^2y$	16. -1 -1	17. $-5\,m^3d^2$ $-2\,m^{10}cd^3$
18. $4\,a^2$ -1	19. $10\,m^4n^3$ $-3\,n^2m^3$	20. $-p^2q$ ap^2q^3	21. $-2\,a^2m^3n^4$ $8\,b^5n^6m^7$
22. $2\,a^{m+1}$ $3\,a^2$	23. $-2\,a^rb^{2z}$ $7\,a^{3r}b^{4z}$	24. $-x^ny^n$ xy	25. $4\,x^{n-1}$ $-2\,x^{n+1}$
26. $5\,y$ $3\,y^{n-2}$	27. $a^2b^3x^3y^{n-2}$ $a^nb^{n-3}y^2$	28. $-x^{1-n}$ $-x^n$	29. $a^{n-1}b^{n-2}c^3$ $a^{n+1}b^2c^{n-1}$
30. $-5\,x^n$ x	31. $-xyz$ $x^{b-1}z^{n+4}$	32. ab^2c^3 $d^mb^{2n}c$	33. $2^{5-n}z^{b-a}$ $2^{n-3}z^{2a-b}$
34. $-a^m$ $-a^n$	35. $-4\,a^rb^s$ $-3\,a^{r-2}b^s$	36. y^{n-m} y^{m-n+1}	37. $-x^{n-1}y^{n-2}$ $-xy$
38. d^{s-n} d^{2s+n}	39. $8\,r^{x-y}s^b$ $3\,r^ys^{a-2b}$	40. z^{r+s-3} z^3	41. $m^an^cb^2y^a$ $m^bn^db^xy^{b-a}$

90. Product of several monomials.

By the law of signs, $-a\cdot-b=+ab$;

$$-a\cdot-b\cdot-c=+ab\cdot-c=-abc;$$

$$-a\cdot-b\cdot-c\cdot-d=-abc\cdot-d=+abcd;\text{ etc.}$$

The product of an even number of negative factors is positive; of an odd number of negative factors, negative.

Positive factors do not affect the sign of the product (§ 83).

EXERCISES

91. Find the products indicated:

1. $(-1)(-1)(-1)(-1)$.
2. $(-2)(-ab)(-3a^2)$.
3. $(-a^2x)(4bx)(-5a^2)$.
4. $(-2xy)(-3xy)(5x^2y)(-xy^2)$.
5. $(-4bc)b(-3c^2)c(-b)(-bc)$.
6. $(-2^3)(-2^4)(5\cdot2^2)(-5^2\cdot2)$.
7. Find the product of 2^n, 2^{n-1}, and 2^{n+1}. Test the correctness of your answer when n represents 3.

92. To multiply a polynomial by a monomial.

$$\begin{array}{r} 43 \\ 2 \\ \hline 86 \end{array}\qquad \begin{array}{r} 40+3 \\ 2 \\ \hline 80+6 \end{array}\qquad \begin{array}{r} 4t+3 \\ 2 \\ \hline 8t+6 \end{array}\qquad \begin{array}{r} 5a-2b \\ 3 \\ \hline 15a-6b \end{array}$$

In general, $a(x+y+z)=ax+ay+az$. That is,

93. *The product of a polynomial by a monomial is equal to the algebraic sum of the partial products obtained by multiplying each term of the polynomial by the monomial.*

This is the **distributive law** for multiplication.

EXERCISES

94. Multiply:

1. $3x^2-2xy$ by $5xy^2$.
2. $3a^3-6a^2b$ by $-2b$.
3. $m^2n^3-3mn^4$ by $2mn$.
4. $p^2q^2-2pq^3$ by $-pq$.
5. $4a^2-5b^2c-c^3$ by abc^2.
6. $-2ac+4ax$ by $-5acx$.

Perform the multiplications indicated:

7. $a^2bc(3a^4 - 4a^3b - 5a^2b^2 + 2ab^3 - 16b^4)$.

8. $2xy(5x^2 - 10xy - 36y^2 - 5x + 5y + 120)$.

9. $5m^3(16m^3 - 20m^2n + 13mn^2 - 25n^3)$.

10. $abc(a^2b^2 - 2a^2c^2 - 2b^2c^2 - a^4 - 4b^4 - c^4 - 5abc)$.

11. $-bc(b^4 + c^4 - b^3 - c^3 + b^2c^2 - 4b^2c + 8bc^2 - 2bc)$.

12. $m^an^3(m^4 - 5m^3n^b - 16m^2n^{2b} + 24mn^{3b} - n^{4b})$.

13. $x^{n-3}y^{m+4}(x^3y^{m-3} - 5x^{4-n}y^{m-2} + 10x^{5-n}y^{m-1} - 5x^{4-2n}y^{2-m} + x^{5-3n}y^{3-2m})$.

95. To multiply a polynomial by a polynomial.

EXERCISES

1. Multiply $x + 5$ by $x + 2$; test the result.

	PROCESS	TEST
	$x + 5$	$= 6$ when $x = 1$
	$x + 2$	$= 3$
x times $(x + 5) =$	$x^2 + 5x$	
2 times $(x + 5) =$	$2x + 10$	
$(x + 2)$ times $(x + 5) =$	$x^2 + 7x + 10$	$= 18$

TEST. — The product must equal the multiplicand multiplied by the multiplier, regardless of what value x may represent. To test the result, therefore, we may assign to x any value we choose and observe whether, for that value, *product obtained* = *multiplicand* × *multiplier*. When $x = 1$, multiplicand $= 6$, multiplier $= 3$, and $x^2 + 7x + 10 = 18$; since $6 \times 3 = 18$, it may be assumed that $x^2 + 7x + 10$ is the correct product.

RULE. — *Multiply the multiplicand by each term of the multiplier and find the algebraic sum of the partial products.*

2. Multiply $x + 4$ by $x + 6$; test the result when $x = 1$.

3. Multiply $x - 1$ by $x - 2$; test the result when $x = 5$.

4. Multiply $2x + 3$ by $4x - 1$; test the result when $x = 1$.

5. Multiply $x^2 + x + 1$ by $x - 1$; test the result when $x = 2$.

6. Multiply $2a-b+c$ by $3a+b$; test the result when $a=1$, $b=1$, and $c=1$.

In like manner the multiplication of any two literal expressions may be tested arithmetically by assigning any values we please to the letters.

While it is usually most convenient to substitute $+1$ for each letter, since this may be done readily by adding the numerical coefficients, the student should bear in mind that this really tests the coefficients and not necessarily the exponents, for any power of 1 is 1.

Multiply, and test each result:

7. $2x+3$ by $x+2$.
8. $4x+1$ by $3x+4$.
9. $5n-1$ by $4n+5$.
10. $h+2k$ by $3h-k$.
11. $3l+5t$ by $2l+6t$.
12. $4y-6b$ by $2y+b$.
13. $2b+5c$ by $5b-2c$.
14. $ab-15$ by $ab+10$.
15. $ax+by$ by $ax-by$.
16. a^2-ay+y^2 by $a+y$.

An indicated product is said to be **expanded** when the multiplication is performed.

Expand, and test each result:

17. $(x+y)(x+y)$.
18. $(c^3+d^3)(c^3+d^3)$.
19. $(3a+b)(3a+b)$.
20. $(3a+b)(3a-b)$.
21. $(2a^2+b)(2a^2-b)$.
22. $(x^n+y^n)(x^n+y^n)$.
23. $(x^n+y^n)(x^n-y^n)$.
24. $(3ax+2by)(3ax+2by)$.
25. $(3ax+2by)(3ax-2by)$.
26. $(a+b+c)(a+b-c)$.

Multiply, and test each result:

27. $2a^2-3b^2-ab$ by $3a^2-4ab-5b^2$.
28. $5x-5x^2+10$ by $12-30x+2x^2$.
29. $3n^2+3m^2+mn$ by $m^3-2mn^2+m^2n$.
30. $4y^2-10+2y$ by $2y^2-3y+5$.
31. $4x-3x^2+2x^3$ by $3x-10x^2+10$.

Multiply, and test each result:

32. $a^5 + a^4 + 4a^3 - a^2 + a$ by $a + 1$.

33. $31x^3 - 27x^2 + 9x - 3$ by $3x + 1$.

34. $4x^3 - 3x^2y + 5xy^2 - 6y^3$ by $5x + 6y$.

35. $a + b + c + d$ by $a - b - c + d$.

36. $a^2 + b^2 + c^2 - ab - ac - bc$ by $a + b + c$.

37. $m^8 - m^6n^3 + m^4n^6 - m^2n^9 + n^{12}$ by $m^2 + n^3$.

Expand:

38. $(\frac{1}{2}a + \frac{1}{3}b)(\frac{1}{2}a - \frac{1}{3}b)$.

39. $(\frac{2}{3}x + \frac{1}{4}y)(\frac{3}{4}x + \frac{1}{2}y)$.

40. $(\frac{1}{3}m - \frac{1}{2}n)(\frac{3}{2}m - \frac{2}{5}n)$.

41. $(\frac{1}{4}b^2 - \frac{1}{3}b + \frac{1}{2})(\frac{1}{3}b - \frac{1}{2})$.

42. $(\frac{3}{4}x^2 - xy + \frac{4}{3}y^2)(\frac{4}{3}x + \frac{3}{4}y)$.

43. $(\frac{2}{3}n^3 - \frac{1}{2}n^2 + \frac{3}{4}n - \frac{1}{4})(3n + 4)$.

44. $(a + b)(a - b)(a + b)(a - b)$.

45. $(a - b)(a + b)(a^2 + b^2)(a^4 + b^4)$.

46. $(2a + 3b + 5c)(2a + 3b - 5c)$.

47. $(5m - 2n + x)(5m - 2n - x)$.

48. $(a^m - b^n)(a^m + b^n)(a^{2m} + b^{2n})$.

49. $(x^n - nx^{n-1}y + \frac{1}{2}nx^{n-2}y^2)(x + y)$.

Multiply:

50. $ax^{2n} + ay^{2n}$ by $ax^{2n} - ay^{2n}$.

51. $ax^{n-1} + y^{n-1}$ by $3ax^{n-1} + 2y^{n-1}$.

52. $x^{2n} + 2x^ny^n + y^{2n}$ by $x^{2n} - 2x^ny^n + y^{2n}$.

53. $a^{6n} + a^{4n}b^{2c} + a^{2n}b^{4c} + b^{6c}$ by $a^{2n} - b^{2c}$.

54. $m^{x+1}n^{x-1} + m^{x-1}n^{x+1} + 1$ by $m^{x-1}n^{x+1} - m^{x+1}n^{x-1} + 1$.

55. $\frac{1}{2}z^{2a+1} - \frac{1}{2}z^{2a} + \frac{1}{2}z^{2a-1}$ by $2z^{2a-1} + 2z^{2a-2} + 2z^{2a-3}$.

96. When a polynomial is arranged so that in passing from left to right the several powers of some letter are successively *higher* or *lower*, the polynomial is said to be **arranged** according to the **ascending** or **descending** powers, respectively, of that letter.

The polynomial $x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$ *is arranged according to the ascending* powers of y and the descending powers of x.

97. When polynomials are arranged according to the ascending or the descending powers of some letter, processes may often be abridged by using the **detached coefficients.**

EXERCISES

98. 1. Expand $(2x^4 - 3x^3 + 3x + 1)(3x + 2)$.

FULL PROCESS

$$\begin{array}{llllll}
2x^4 & -3x^3 & & +3x & +1 & \\
3x & +2 & & & & \\
\hline
6x^5 & -9x^4 & & +9x^2 & +3x & \\
 & 4x^4 & -6x^3 & & +6x & +2 \\
\hline
6x^5 & -5x^4 & -6x^3 & +9x^2 & +9x & +2
\end{array}$$

DETACHED COEFFICIENTS

$$\begin{array}{llllll}
2 & -3 & +0 & +3 & +1 & \\
3 & +2 & & & & \\
\hline
6 & -9 & +0 & +9 & +3 & \\
 & 4 & -6 & +0 & +6 & +2 \\
\hline
6x^5 & -5x^4 & -6x^3 & +9x^2 & +9x & +2
\end{array}$$

Since the second power of x is wanting in the first factor, the term, if it were supplied, would be $0x^2$. Therefore the detached coefficient of the term is 0. The detached coefficients of missing terms should be supplied to prevent confusion in placing the coefficients in the partial products and to avoid errors in writing the letters in the result.

Arrange properly and expand, using detached coefficients:

2. $(x + x^3 + 1 + x^2)(x - 1)$.

3. $(x^3 + 10 - 7x - 4x^2)(x - 2)$.

4. $(14 - 9x - 6x^2 + x^3)(x + 1)$.

5. $(a^3 + 4a^2 - 11a - 30)(a - 1)$.

6. $(4a^2 - 2a^3 - 8a + a^4 - 3)(2 + a)$.

7. $(2m - 3 + 2m^3 - 4m^2)(2m - 3)$.

8. $(x + x^2 - 5)(x^2 - 3 - 2x)$.

9. $(b^2 + 5b - 4)(-4 + 2b^2 - 3b)$.

10. $(4n^3 + 6 - 2n^4 + 16n - 8n^2 + n^5)(n + 2)$.

11. $(7 - 6x + 5x^2 - 4x^3 + 3x^4 - 2x^5 + x^6)(x^2 + 2x + 1)$.

12. $(1 + x + 4x^2 + 10x^3 + 46x^5 + 22x^4)(2x^2 + 1 - 3x)$.

99. Multiply $a^3+2a^2b+2ab^2+b^3$ by a^2+ab+b^2.

PROCESS TEST

$$\begin{array}{llllll|r}
a^3+2a^2b+2ab^2+b^3 & & & & & & =6\\
a^2+ab+b^2 & & & & & & =\underline{3}\\
\hline
a^5+2a^4b+2a^3b^2+a^2b^3 & & & & & & \\
\quad a^4b+2a^3b^2+2a^2b^3+ab^4 & & & & & & \\
\quad\quad a^3b^2+2a^2b^3+2ab^4+b^5 & & & & & & \\
\hline
a^5+3a^4b+5a^3b^2+5a^2b^3+3ab^4+b^5 & & & & & & =18
\end{array}$$

When each letter of an expression is given the value 1, the expression is equal to the sum of its numerical coefficients. The test on the right of the process, then, tests the signs and coefficients in the product, but not the exponents.

To test the exponents in the product, observe that each term of the multiplicand contains *three literal factors*, as *aaa*, *aab*, etc., or is of the *third degree;* also that each term of the multiplier is of the *second degree.* Therefore, each term of the product should be of the *fifth degree.*

When all the terms of an expression are of the same degree, the expression is called a **homogeneous expression.**

The multiplicand in the process is a homogeneous expression of the third degree; the multiplier is a homogeneous expression of the second degree; and the product is a homogeneous expression of the fifth degree.

As a further test observe that the multiplicand involves a and b in exactly the same way, b corresponding to a, b^2 to a^2, and b^3 to a^3, so that if b and a were interchanged the multiplicand would not be changed, except in the order of terms. Such an expression is said to be **symmetrical** in a and b. Since both multiplicand and multiplier are symmetrical in a and b, the product should be symmetrical in a and b.

100. *The product of two homogeneous expressions is a homogeneous expression.*

If two expressions are symmetrical in the same letters, their product is symmetrical in those letters.

EXERCISES

101. Expand, and test each result:

1. $(a+b+c)(a+b+c)$.
2. $(a^2-ab+b^2)(a^2+ab+b^2)$.
3. $(a^3+3a^2b+3ab^2+b^3)(a+b)$.
4. $(x^4-x^3y+x^2y^2-xy^3+y^4)(x+y)$.
5. $(a^2+b^2+c^2+d^2)(a^2-b^2+c^2-d^2)$.
6. $(ab+bc+cd+bd)(ab+bc-cd-bd)$.
7. $(x^2-xy+y^2+x+y+1)(x+y+1)$.
8. $(a^3+3a^2b+3ab^2+b^3)(a^2+2ab+b^2)$.
9. $(a^2-ab-ac+b^2-bc+c^2)(a+b+c)$.

Addition, Subtraction, Multiplication

EXERCISES

102. 1. Simplify $a^2+a(b-a)-b(2b-3a)$.

SOLUTION. — The expression indicates the algebraic sum of a^2, $a(b-a)$, and $-b(2b-3a)$. Expanding, $a(b-a)=ab-a^2$, and $-b(2b-3a)=-2b^2+3ab$. Therefore, writing the terms in order with their proper signs, $a^2+a(b-a)-b(2b-3a)=a^2+ab-a^2-2b^2+3ab=4ab-2b^2$.

Simplify:

2. $x^2+x(y-x)$.
3. $c^2-c(c-d)$.
4. $5-2(x-3)$.
5. $x^2-y^2-(x-y)^2$.
6. $c(a-b)-c(a+b)$.
7. $a^3-b^3-3ab(a-b)$.
8. $-2(x^2y-xy^2)-5(xy^2-x^2y)$.
9. $(3a-2)(2a-3)-6(a-2)(a-1)$.
10. $8x^3-(4x^2-2xy+y^2)(2x+y)$.
11. $(3m-1)(m+2)-3m(m+3)+2(m+1)$.
12. $(a-b)^2-2(a^2-b^2)-2a(-a-b)-4b^2$.

13. $4(ax - bx + cx - dx) - 3(ax + bx - cx - dx)$.

14. $(x+1)(x+2) - 2(x-1)(x-2) + 4(x+2)(x+3)$.

15. $(x^2 + 2\,xy + y^2)(x^2 - 2\,xy + y^2) - (x^2 + y^2)(x^2 + y^2)$.

16. $b^4 + (a^2 - ab + b^2)(a^2 + b^2) - (a^3 - b^3)(a + 2\,b)$.

17. $y^3 - [2\,x^3 - xy(x-y) - y^3] + 2(x-y)(x^2 + xy + y^2)$.

103. Numerical substitution.

EXERCISES

1. When $a = -2$, $b = 3$, $c = 4$, find the value of

$$a^2 - (a-c)(b+c) + 2\,b.$$

SOLUTION.
$$\begin{aligned} a^2 - (a-c)(b+c) + 2\,b &= (-2)^2 - (-2-4)(3+4) + 2 \cdot 3 \\ &= 4 - (-6)(7) + 6 \\ &= 4 - (-42) + 6 \\ &= 4 + 42 + 6 = 52. \end{aligned}$$

When $x = 3$, $y = -4$, $z = 0$, $m = 6$, $n = 2$, find the value of:

2. $m(x-y) + z^2$.

3. $z + m^2 - (y^3 - 1)$.

4. $x^2 - y^2 - m^2 + n^2$.

5. $(x+y)(m-n) + 3z$.

6. $(m+x)^2 - (n-y)^2 - y^4$.

7. $xyz - n(x-m)^3 - (nx)^3$.

8. $3\,m(m-n) + 4\,n(y-x) - 7(y+z)$.

9. $\frac{1}{2}(y - 2\,n) - \frac{3}{4}(n - 2\,y)(3\,y - 4\,n)$.

10. $x^2y^2(m-n)^2(m+n) + (m+n)^2(m-n)$.

11. $(x-y)^2 - xy(x-y)(x+y)(x^2+y^2)$.

12. $3\,m(x-y-n)^2 - (y-n-x)(n-x-y)$.

13. $(2\,x+y)^n - (x^2 - 2\,y)^x - (m+n)^2(x+y+z)^3$.

14. $(x+y+z)^2 - xy(y+z-x)(x+z-y) - z(x+y-z)$.

15. $(m+n+x)^n - (m+n-x)^n - (m-n+x)^n(-m+n+x)^n$.

16. Show that $(a-b+c)^2 = a^2 + b^2 + c^2 - 2\,ab + 2\,ac - 2\,bc$, when $a = 1$, $b = 2$, and $c = 3$; when $a = 4$, $b = 2$, and $c = -1$.

17. By substituting numbers for a, b, and c, show that $(a+b)(b+c)(c+a) + abc = (a+b+c)(ab+bc+ac)$.

SPECIAL CASES IN MULTIPLICATION

104. The square of the sum of two numbers.

Show by actual multiplication that

$$(a+b)(a+b)=a^2+2\,ab+b^2;$$

also that $$(x+y)(x+y)=x^2+2\,xy+y^2.$$

105. PRINCIPLE. — *The square of the sum of two numbers is equal to the square of the first number, plus twice the product of the first and second, plus the square of the second.*

Since $5\,a^3\times 5\,a^3=25\,a^6$, $3\,a^4b^5\times 3\,a^4b^5=9\,a^8b^{10}$, etc., it is evident that in squaring a number the exponents of literal factors are doubled.

EXERCISES

106. Expand by inspection, and test each result:

1. $(p+q)(p+q)$.
2. $(r+s)(r+s)$.
3. $(a+x)(a+x)$.
4. $(x+4)(x+4)$.
5. $(a+6)(a+6)$.
6. $(y+7)(y+7)$.
7. $(z+1)(z+1)$.
8. $(5\,x+z)^2$.
9. $(2\,a+x)^2$.
10. $(ab+cd)^2$.
11. $(5\,x+2\,y)^2$.
12. $(7\,z+3\,c)^2$.
13. $(3\,b+10\,x)^2$.
14. $(3\,xy^2+4\,x^2y)^2$.
15. $(a^3+b^3)^2$.
16. $(a^5+b^5)^2$.
17. $(a^n+b^n)^2$.
18. $(x^m+y^n)^2$.
19. $(3\,a^2+5\,b^3)^2$.
20. $(1+2\,a^3b)^2$.
21. $(x^{n-1}+y^{n-1})^2$.

107. The square of the difference of two numbers.

Show by actual multiplication that

$$(a-b)(a-b)=a^2-2\,ab+b^2;$$

also that $$(x-y)(x-y)=x^2-2\,xy+y^2.$$

108. PRINCIPLE. — *The square of the difference of two numbers is equal to the square of the first number, minus twice the product of the first and second, plus the square of the second.*

EXERCISES

109. Expand by inspection, and test each result:

1. $(x-m)(x-m)$.
2. $(m-n)(m-n)$.
3. $(x-6)(x-6)$.
4. $(p-8)(p-8)$.
5. $(q-7)(q-7)$.
6. $(a-c)(a-c)$.
7. $(a-x)(a-x)$.
8. $(x-1)(x-1)$.
9. $(2a-x)^2$.
10. $(3m-n)^2$.
11. $(4x-y)^2$.
12. $(m-4n)^2$.
13. $(p-3q)^2$.
14. $(a-7b)^2$.
15. $(4a-3)^2$.
16. $(5x-4)^2$.
17. $(3x-2)^2$.
18. $(2x-5y)^2$.
19. $(5m-3n)^2$.
20. $(3p-5q)^2$.
21. $(a^n-b^n)^2$.
22. $(x^m-y^n)^2$.
23. $(x^{m-1}-y^{n-1})^2$.
24. $(mx^m-ny^n)^2$.

110. The square of any polynomial.

Show by actual multiplication that

$$(a+b+c)^2=a^2+b^2+c^2+2ab+2ac+2bc;$$

also that
$$(a+b+c+d)^2=a^2+b^2+c^2+d^2+2ab+2ac$$
$$+2ad+2bc+2bd+2cd.$$

Similarly, by squaring any polynomial by multiplication, it will be observed that:

111. PRINCIPLE. — *The square of a polynomial is equal to the sum of the squares of the terms and twice the product of each term by each term, taken separately, that follows it.*

When some of the terms are negative, some of the double products will be negative, but the squares will always be positive. For example, since $(-b)^2=+b^2$, $(a-b+c)^2=a^2+(-b)^2+c^2+2a(-b)+2ac+2(-b)c$ $=a^2+b^2+c^2-2ab+2ac-2bc$.

EXERCISES

112. Expand by inspection, and test each result:

1. $(x+y+z)^2$.
2. $(x+y-z)^2$.
3. $(x-y-z)^2$.
4. $(x-y+z)^2$.
5. $(x+y-3z)^2$.
6. $(x-y+3z)^2$.

Expand by inspection, and test each result:

7. $(a-2b+c)^2$.
8. $(2a-b-c)^2$.
9. $(b-2a+c)^2$.
10. $(ax-by+cz)^2$.
11. $(qb-pc-rd)^2$.
12. $(ac-bd-de)^2$.
13. $(3x-2y+4z)^2$.
14. $(2a-5b+3c)^2$.
15. $(2m-4n-r)^2$.
16. $(12-2x+3y)^2$.
17. $(a+m+b+n)^2$.
18. $(a-m+b-n)^2$.

113. The product of the sum and difference of two numbers.

Let a and b represent any two numbers, $a+b$ their sum, and $a-b$ their difference.

Show by actual multiplication that

$$(a+b)(a-b)=a^2-b^2.$$

114. Principle. — *The product of the sum and difference of two numbers is equal to the difference of their squares.*

EXERCISES

115. Expand by inspection, and test each result:

1. $(x+y)(x-y)$.
2. $(a+c)(a-c)$.
3. $(p+q)(p-q)$.
4. $(p+5)(p-5)$.
5. $(x+1)(x-1)$.
6. $(x^2+1)(x^2-1)$.
7. $(x^3+1)(x^3-1)$.
8. $(x^4-1)(x^4+1)$.
9. $(x^5-1)(x^5+1)$.
10. $(x^n+y^3)(x^n-y^3)$.
11. $(ab+5)(ab-5)$.
12. $(cd+d^2)(cd-d^2)$.
13. $(2x+3y)(2x-3y)$.
14. $(3m+4n)(3m-4n)$.
15. $(12+xy)(12-xy)$.
16. $(3m^2n-b)(3m^2n+b)$.
17. $(ab+cd)(ab-cd)$.
18. $(2x^3+5y^2)(2x^3-5y^2)$.
19. $(3x^6+2y^6)(3x^6-2y^6)$.
20. $(2a^2+2b^2(2a^2-2)b^2)$.

21. $(-5n-b)(-5n+b)$.

22. $(-x-2y)(-x+2y)$.

23. $(-4-3a)(-4-3a)$.

24. $(x^{m-1}+y^{n+1})(x^{m-1}-y^{n+1})$.

25. $(3x^m+7y^n)(3x^m-7y^n)$.

26. $(5a^3b^2+2x^c)(5a^3b^2-2x^c)$.

One or both numbers may consist of more than one term.

27. Expand $(a+m-n)(a-m+n)$.

SOLUTION

$$a+m-n=a+(m-n).$$

$$a-m+n=a-(m-n).$$

$$\therefore [a+m-n][a-m+n]=[a+(m-n)][a-(m-n)]$$

Prin., § 114, $\quad = a^2-(m-n)^2$

§ 108, $\quad = a^2-(m^2-2mn+n^2)$

$$= a^2-m^2+2mn-n^2.$$

Expand:

28. $(a+x-y)(a-x+y)$.

29. $(x+c-d)(x-c+d)$.

30. $(r+p-q)(r-p+q)$.

31. $(r+p+q)(r-p-q)$.

32. $(x+b+n)(x-b-n)$.

33. $(y+c+d)(y+c-d)$.

34. $(a+x+y)(a+x-y)$.

35. $(x^2+2x+1)(x^2+2x-1)$.

36. $(x^2+2x-1)(x^2-2x+1)$.

37. $(x^2+3x-2)(x^2-3x+2)$.

38. $(m^4-2m^2+1)(m^4+2m^2+1)$.

39. $(2x+3y-4z)(2x+3y+4z)$.

40. $(2x^2-xy+3y^2)(2x^2+xy-3y^2)$.

41. $(x^2+xy+y^2)(x^2-xy+y^2)$.

42. $[(a+b)+(c+d)][(a+b)-(c+d)]$.

43. $(a+b+x+y)(a+b-x-y)$.

44. $(a+b+m-n)(a+b-m+n)$.

45. $(x-m+y-n)(x-m-y+n)$.

46. $(p-q+r+s)(p-q-r-s)$.

47. $(a-m-b-n)(a+m-b+n)$.

116. The product of two binomials that have a common term.

Let $x+a$ and $x+b$ represent any two binomials having a common term, x. Multiplying $x+a$ by $x+b$,

$$\begin{array}{l} x+a \\ \underline{x+b} \\ x^2+ax \\ \qquad\underline{bx+ab} \\ x^2+(a+b)x+ab \end{array}$$

117. Principle. — *The product of two binomials having a common term is equal to the sum of the square of the common term, the product of the sum of the unlike terms and the common term, and the product of the unlike terms.*

EXERCISES

118. 1. Expand $(x+2)(x+5)$ and test the result.

Solution. — The square of the common term is x^2;
the sum of 2 and 5 is 7;
the product of 2 and 5 is 10;
$\therefore (x+2)(x+5)=x^2+7x+10.$

Test. — If $x=1$, we have $3\cdot 6=1+7+10$, or $18=18$.

2. Expand $(a+1)(a-4)$ and test the result.

Solution. — The square of the common term is a^2;
the sum of 1 and -4 is -3;
the product of 1 and -4 is -4;
$\therefore (a+1)(a-4)=a^2-3a-4.$

Test. — If $a=4$, we have $5\cdot 0=16-12-4$, or $0=0$.

3. Expand $(n-2)(n-3)$ and test the result.

Solution. — The square of the common term is n^2;
the sum of -2 and -3 is -5;
the product of -2 and -3 is 6;
$\therefore (n-2)(n-3)=n^2-5n+6.$

Test. — If $n=3$, we have $1\cdot 0=9-15+6$, or $0=0$.

Expand by inspection, and test each result:

4. $(x+5)(x+6)$.
5. $(x+7)(x+8)$.
6. $(x-7)(x+8)$.
7. $(x+7)(x-8)$.
8. $(x-5)(x-4)$.
9. $(x-3)(x-2)$.
10. $(x-5)(x-1)$.
11. $(x+5)(x+8)$.
12. $(p-4)(p+1)$.
13. $(r-3)(r-1)$.
14. $(n-8)(n-12)$.
15. $(n-6)(n+15)$.
16. $(x^2+5)(x^2-3)$.
17. $(x^3-7)(x^3+6)$.
18. $(x^n-5)(x^n+4)$.
19. $(x^n-a)(x^n-b)$.
20. $(y-2a)(y+3b)$.
21. $(z-4a)(z+3a)$.
22. $(2x+5)(2x+3)$.
23. $(2x-7)(2x+5)$.
24. $(3y-1)(3y+2)$.
25. $(4x^2+1)(4x^2-7)$.
26. $(ab-6)(ab+4)$.
27. $(x^2y^2-a)(x^2y^2+2a)$.
28. $(3xy+y^2)(y^2-xy)$.
29. $(\overline{x+y}-1)(\overline{x+y}+2)$.
30. $(\overline{x-y}-2)(\overline{x-y}-8)$.
31. $(\overline{x^2+x}-1)(\overline{x^2+x}+3)$.

By an extension of the method given above, the product of any two binomials having similar terms may be written.

32. Expand $(2x-5)(3x+4)$.

PROCESS

$$\begin{array}{l} 2x-5 \\ 3x+4 \\ \hline 6x^2-7x-20 \end{array}$$

EXPLANATION. — The product must have a term in x^2, a term in x, and a numerical, or **absolute**, term. The x^2 term is the product of $2x$ and $3x$; the x term is the sum of the partial products $-5 \cdot 3x$ and $2x \cdot 4$, called the **cross-products**; and the absolute term is the product of -5 and 4.

The process should not be used except as an aid in explanation.

Expand by inspection, and test each result:

33. $(2x+5)(3x+4)$.
34. $(3x-2)(2x-3)$.
35. $(3a-4)(4a+3)$.
36. $(3x-y)(x-3y)$.
37. $(2a+5b)(5a+2b)$.
38. $(7n^2-2p)(2n^2-7p)$.

Algebraic Representation

119. 1. Express in the shortest way the sum of five x's; the product of five x's.

2. When the multiplicand is x and the multiplier y, express the product in three ways.

3. Indicate the product when the sum of x, y, and $-d$ is multiplied by xy.

4. How much will a man whose wages are a dollars per day earn in b days? in c days? in x days? in a days?

5. If a man earns a dollars per month and his expenses are b dollars per month, how much will he save in a year?

6. Indicate the sum of x and z multiplied by m times the sum of x and y.

7. From x subtract m times the sum of the squares of $(a+b)$ and $(a-b)$.

8. A number x is equal to $(y-c)$ times $(d+c)$. Write the equation.

9. How many seconds are x days $+ c$ hours $+ d$ minutes?

10. Express in cents the interest on y dollars for x years, if the interest for one month is z cents on one dollar.

11. How far can a wheelman ride in a hours at the rate of b miles an hour? How far will he have ridden after a hours, if he stops c hours of the time to rest?

12. How many square rods are there in a square field one of whose sides is $2b$ rods long? $(x-y)$ rods long?

13. What is the number of square rods in a rectangular field whose length is $(a+b)$ rods and width $(a-b)$ rods?

14. A fence is built across a rectangular field so as to make the part on one side of the fence a square. If the field is a rods long and b rods wide, what is the area of each part?

15. Represent $(a-b)$ times the number whose tens' digit is x and units' digit y.

Equations and Problems

120. 1. Given $5(2x-3)-7(3x+5)=-72$, to find the value of x.

SOLUTION

$$5(2x-3)-7(3x+5)=-72.$$

Expanding, $\quad 10x-15-21x-35=-72.$

Transposing, $\quad 10x-21x=15+35-72.$

Uniting terms, $\quad -11x=-22.$

Multiplying by -1, $\quad 11x=22.$

$$\therefore x=2.$$

VERIFICATION. — Substituting 2 for x in the given equation,

$$5(4-3)-7(6+5)=-72.$$
$$5-77=-72.$$

Hence, 2 is a true value of x.

Find the value of x, and verify the result, in:

2. $2=2x-5-(x-3)$.
3. $10x-2(x-3)=22$.
4. $1=5(2x-4)+5x+6$.
5. $7(5-3x)=3(3-4x)-1$.
6. $2(x-5)+7=x+30-9(x-3)$.
7. $5+7(x-5)=15(x+2-36)$.
8. $(x-2)(x-2)=(x-3)(x-3)+7$.
9. $(x-4)(x+4)=(x-6)(x+5)+25$.
10. $4x^3-4(x^3-x^2+x-2)=4x^2$.
11. $7(2x-3b)=2b-3(2x+b)$.
12. $3(2b-4x)-(x-b)=-6b$.
13. $4x-x^2=x(2-x)+2a$.
14. $3(x-a-2b)=3b$.
15. $5b=3(2x-b)-4b$.
16. $x^2-(2x+3)(2x-3)+(2x-3)^2=(x+9)(x-2)-2$.
17. $3(4-x)^2-2(x+3)=(2x-3)^2-(x+2)(x-2)+1$.
18. $20(2-x)+3(x-7)-2[x+9-3\{9-4(2-7)\}]=23$.

19. Jamaica one year exported 16,000,000 bunches of bananas. The number of bunches sent to the United States, less 600,000, was 10 times the number sent to all other countries. How many bunches were exported to the United States?

20. An electric light company expended 60¢ for every dollar of income. Fuel cost 5¢ less than $\frac{1}{3}$ as much as other things. What was the cost of fuel per dollar of income?

21. Police protection in a large city, one year, cost \$6,500,000 less than education. The total expenditure of the city, \$98,100,000, was \$6,600,000 more than 3 times these two items. What sum was devoted to education? to police protection?

22. Cherries brought 2¢ more per pound in 8-lb. boxes than in 5-lb. boxes, and a 5-lb. box sold for 2¢ less than $\frac{1}{2}$ as much as an 8-lb. box. What was each price per pound?

23. Upon the floor of a room 4 feet longer than it is wide is laid a rug whose area is 112 square feet less than the area of the floor. There are 2 feet of bare floor on each side of the rug. What is the area of the rug? of the floor?

24. The number of hundred violets sold by a florist during December and January was 240. The price per hundred was \$2 in December and \$$1\frac{1}{2}$ in January, and the total sum received was \$405. How many hundred violets were sold each month?

25. A party of 8 traveled second class from London to Paris for \$5.70 less than twice the amount paid by a party of 3 traveling first class. If a first-class ticket cost \$4.15 more than a second-class ticket, find the price of each.

26. If, in coaling the British battleship *Terrible* on one occasion, the amount loaded per hour had been 63 tons less, the time taken would have been $12\frac{1}{2}$ hours. If the amount per hour had been 13 tons less, the time would have been 10 hours. How many tons were put on board per hour?

DIVISION

121. In multiplication two numbers are given and their product is to be found. The *inverse* process, finding one of two numbers when their product and the other number are given, is called **division**.

$$10 \div 2 = 5, \text{ and } D \div d = q$$

are inverses of

$$5 \times 2 = 10, \text{ and } q \times d = D.$$

The **dividend** corresponds to the product, the **divisor** to the multiplier, and the **quotient** to the multiplicand.

Hence, the *quotient* may be defined as *that number which multiplied by the divisor produces the dividend.*

In general, the quotient of any two numbers, as a divided by b, indicated by $a \div b$, or $\frac{a}{b}$, is defined by the relation

$$\frac{a}{b} \times b = a.$$

122. Sign of the quotient.

The following are direct consequences of the law of signs for multiplication (§ 84) and the definition of quotient:

$$(+a)(+b) = +ab; \quad \therefore \ +ab \div (+b) = +a.$$
$$(-a)(+b) = -ab; \quad \therefore \ -ab \div (+b) = -a.$$
$$(+a)(-b) = -ab; \quad \therefore \ -ab \div (-b) = +a.$$
$$(-a)(-b) = +ab; \quad \therefore \ +ab \div (-b) = -a.$$

123. Law of Signs for Division.—*The sign of the quotient is + when the dividend and divisor have* **like signs,** *and − when they have* **unlike signs.**

EXERCISES

124. 1. Divide each of the following numbers by 2.

6, −6, 10, −10, 14, −12, −18, 22, −8.

2. Divide each of the foregoing numbers by −2.

Perform the indicated divisions:

3. $7\overline{)-14}$. 4. $-3\overline{)15}$. 5. $-3\overline{)-12}$. 6. $-1\overline{)9}$.

7. $4 \div (-4)$. 8. $22 \div (-2)$. 9. $-1 \div (-1)$. 10. $-6 \div 3$.

11. $\frac{36}{4}$. 12. $\frac{28}{-7}$. 13. $\frac{-42}{6}$. 14. $\frac{-20}{-5}$.

125. To divide a monomial by a monomial.

Since $7\,a \times 3\,a^4 = 21\,a^5$,

by def. of quotient, $21\,a^5 \div 3\,a^4 = 7\,a$.

The quotient may be obtained, as in arithmetic, by removing equal factors from dividend and divisor, thus:

$$\frac{21\,a^5}{3\,a^4} = \frac{7 \cdot \cancel{3} \cdot \cancel{a} \cdot \cancel{a} \cdot \cancel{a} \cdot \cancel{a} \cdot a}{\cancel{3} \cdot \cancel{a} \cdot \cancel{a} \cdot \cancel{a} \cdot \cancel{a}} = 7\,a,$$

or

$$\frac{21\,a^5}{3\,a^4} = \frac{21}{3} a^{5-4} = 7\,a^1 = 7\,a.$$

126. Law of Coefficients for Division. — *The coefficient of the quotient is equal to the coefficient of the dividend divided by the coefficient of the divisor.*

127. Law of Exponents, or Index Law, for Division. — *The exponent of a number in the quotient is equal to its exponent in the dividend minus its exponent in the divisor.*

Since a number divided by itself equals 1, $a^5 \div a^5 = a^{5-5} = a^0 = 1$; that is, a number whose exponent is 0 is equal to 1. (Discussed in § 306.)

The law of exponents for division is of general application, but for present *purposes* exponents will be limited to *positive integers.* *The proof* for *positive* integral exponents follows:

Let m and n be positive integers, m being greater than n; and let a be any number.

By notation, § 27, $a^m = a \cdot a \cdot a \cdots$ to m factors,

and $a^n = a \cdot a \cdot a \cdots$ to n factors;

$$\therefore \frac{a^m}{a^n} = \frac{a \cdot a \cdot a \cdots \text{to } m \text{ factors}}{a \cdot a \cdot a \cdots \text{to } n \text{ factors}}.$$

Remove equal factors from dividend and divisor. Then,

$$a^m \div a^n = a \cdot a \cdot a \cdots \text{to } (m - n) \text{ factors}$$

by notation, $= a^{m-n}$.

EXERCISES

128. 1. $\begin{array}{l}5\overline{)5^3}\\ \quad 5^2\end{array}$. 2. $\begin{array}{l}7\,c^2d^4\overline{)-35\,c^4d^5}\\ \qquad\quad -\ 5\,c^2d\end{array}$. 3. $\begin{array}{l}-4\,a^3\overline{)-a^5}\\ \qquad\quad \frac{1}{4}\,a^2\end{array}$.

Divide as indicated:

4. $2^2\overline{)2^3}$. 5. $3^4 \div 3^4$. 6. $a^4\overline{)a^{10}}$.

7. $2^2\overline{)2^4}$. 8. $4^5 \div 4^0$. 9. $x^n\overline{)x^m}$.

10. $\dfrac{28\,a^4b^2c}{-4\,abc}$. 11. $\dfrac{-16\,x^3y^3z^3}{4\,xy^2z}$. 12. $\dfrac{\frac{4}{3}\,\pi r^3}{2\,\pi r}$.

13. $\dfrac{20\,a^4b^5y^2}{4\,a^2b^2y^2}$. 14. $\dfrac{-36\,a^4y^2z^3}{-9\,a^4z^2}$. 15. $\dfrac{3\,ab(a+b)^2}{-2(a+b)}$.

16. $\dfrac{4\,a^4b^3c^5}{20\,a^2bc^3}$. 17. $\dfrac{-4\,x^6y^3z^4}{32\,x^4y^2z^3}$. 18. $\dfrac{2\,a^2(x-y)^3}{-a(x-y)^2}$.

129. To divide a polynomial by a monomial.

Since, § 93, $(a+b)x = ax + bx$,

if $ax+bx$ is regarded as the dividend (§ 121) and x as the divisor,

$$(ax + bx) \div x = a + b; \text{ that is,}$$

130. *The quotient of a polynomial by a monomial is equal to the algebraic sum of the partial quotients obtained by dividing each term of the polynomial by the monomial.*

This is the **distributive law** for division.

EXERCISES

131. 1. Divide $4\,a^3b - 6\,a^2b^2 + 4\,ab^3$ by $2\,ab$; by $-2\,ab$.

PROCESS

$$2\,ab\overline{)4\,a^3b - 6\,a^2b^2 + 4\,ab^3}$$
$$2\,a^2 \quad -3\,ab \quad +2\,b^2$$

PROCESS

$$-2\,ab\overline{)\;4\,a^3b - 6\,a^2b^2 + 4\,ab^3}$$
$$-2\,a^2 \quad +3\,ab \quad -2\,b^2$$

Test of Signs. — When the divisor is positive, the signs of the quotient should be *like* those of the dividend. When the divisor is negative, the signs of the quotient should be *unlike* those of the dividend.

Test of Exponents. — Since the sum of the exponents in each term of the dividend is 4, and the sum in each term of the divisor is 2, the sum of the exponents in each term of the quotient should be $4 - 2$, or 2.

Find the quotient:

2. $\dfrac{4\,a^2b^3 - 12\,a^3b^2 + 16\,a^4b}{4\,a^2b}$.

3. $\dfrac{24\,a^6b^2 + 32\,a^5b^3 - 40\,a^4b^4}{8\,a^4b^2}$.

4. $\dfrac{-35\,x^2y^3z^4 + 45\,x^4y^3z^2}{5\,x^2y^2z}$.

5. $\dfrac{-39\,x^3y^4z^6 + 65\,x^2y^5z^7}{-13\,x^2y^4z^6}$.

6. $\dfrac{4\,m^3n - 8\,m^2n^2 + 4\,mn^3}{4\,mn}$.

7. $\dfrac{5\,x^4y - 10\,x^3y^2 + 20\,x^2y^3}{5\,xy}$.

8. $\dfrac{-a - b - c - d - e}{-1}$.

9. $\dfrac{-a + a^2b - a^3c - a^4d + a^5e}{-a}$.

10. $(34\,a^2x^6y^2 - 51\,a^4x^4y^4 - 68\,a^6x^2y^6) \div 17\,a^2x^2y^2$.

11. $(8\,a^7b^3 - 28\,a^6b^4 - 16\,a^5b^5 + 4\,a^4b^6) \div 4\,a^4b^3$.

12. $[a(b-c)^3 - b(b-c)^2 + c(b-c)] \div (b-c)$.

13. $[(x-y) - 3(x-y)^2 + 4\,x(x-y)^3] \div (x-y)$.

14. $(x^a + 2\,x^{a+1} - 5\,x^{a+2} - x^{a+3} + 3\,x^{a+4}) \div x^a$.

15. $(y^{n+1} - 2\,y^{n+2} + y^{n+3} - 3\,y^{n+4} + y^{n+5}) \div y^{n+1}$.

16. $(x^n - x^{n-1} + x^{n-2} - x^{n-3} + x^{n-4} - x^{n-5}) \div x^2$.

17. $(r^{8m}s^{6n} - 3\,r^{6m}s^{8n} - 5\,r^{4m}s^{10n}) \div (-5\,r^{4m}s^{6n})$.

18. $(a^{2x+3}b^{x+2} - a^{2x+2}b^{x+4} + a^{2x+1}b^{x+6}) \div a^{2x}b^{x+1}$.

132. To divide a polynomial by a polynomial.

EXERCISES

1. Divide $3x^2 + 35 + 22x$ by $x + 5$.

	PROCESS		TEST
	$3x^2 + 22x + 35$	$x + 5$	$60 \div 6$
$3x$ times $(x+5)$	$3x^2 + 15x$	$3x + 7$	$= 10$
	$7x + 35$		
7 times $(x + 5)$	$7x + 35$		

EXPLANATION. — For convenience, the divisor is written at the right of the dividend, and both are arranged according to the descending powers of x.

Since the dividend is the product of the quotient and divisor, it is the algebraic sum of all the products formed by multiplying each term of the quotient by each term of the divisor. Therefore, the term of highest degree in the dividend is the product of the terms of highest degree in the quotient and divisor. Hence, if $3x^2$, the first term of the dividend as arranged, is divided by x, the first term of the divisor, the result, $3x$, is the term of highest degree, or the first term, of the quotient.

Subtracting $3x$ times $(x + 5)$ from the dividend, leaves a remainder of $7x + 35$.

Since the dividend is the algebraic sum of the products of each term of the quotient multiplied by the divisor, and since the product of the first term of the quotient multiplied by the divisor has been canceled from the dividend, the remainder, or *new dividend*, is the product of the rest of the quotient multiplied by the divisor.

Proceeding, then, as before we find, $7x \div x = 7$, the next term of the quotient. 7 times $(x + 5)$ equals $7x + 35$. Subtracting, we have no remainder. Hence, all of the terms of the quotient have been obtained, and the quotient is $3x + 7$.

TEST. — Let $x = 1$.

Dividend	$= 3x^2 + 22x + 35$	$= 3 + 22 + 35$	$= 60.$
Divisor	$= x + 5$	$= 1 + 5$	$= 6.$
Quotient should be equal to			10
Quotient	$= 3x + 7$	$= 3 + 7$	$= 10.$

Similarly, the result may be tested by substituting any other value for x, *except such a value* as gives for the result $0 \div 0$, or any number divided *by 0, for reasons* that will be shown in § 547.

Rule. — *Arrange both dividend and divisor according to the ascending or the descending powers of a common letter.*

Divide the first term of the dividend by the first term of the divisor, and write the result for the first term of the quotient.

Multiply the whole divisor by this term of the quotient, and subtract the product from the dividend. The remainder will be a new dividend.

Divide the new dividend as before, and continue to divide in this way until the first term of the divisor is not contained in the first term of the new dividend.

If there is a remainder after the last division, write it over the divisor in the form of a fraction, and add the fraction to the part of the quotient previously obtained.

Divide, and test each result:

2. $x^2 + x - 20$ by $x + 5$.
3. $x^2 + 7x + 12$ by $x + 3$.
4. $l^4 - 6l^2 - 16$ by $l^2 + 2$.
5. $m^2 - 18 - 3m$ by $m - 6$.
6. $x^2 + 15x + 54$ by $x + 6$.
7. $10 - 11x + x^2$ by $x - 10$.
8. $81 + 9a^2 + a^4$ by $a^2 - 3a + 9$.

PROCESS **TEST**

$$
\begin{array}{l|l}
a^4 + 9a^2 + 81 & a^2 - 3a + 9 \\
\underline{a^4 - 3a^3 + 9a^2} & \overline{a^2 + 3a + 9} \\
\quad 3a^3 + 81 & \\
\quad \underline{3a^3 - 9a^2 + 27a} & \\
\qquad 9a^2 - 27a + 81 & \\
\qquad \underline{9a^2 - 27a + 81} &
\end{array}
\qquad
\begin{array}{r}
91 \div 7 \\
= 13
\end{array}
$$

9. $a^4 + 16 + 4a^2$ by $2a + a^2 + 4$.
10. $x^5 - 61x - 60$ by $x^2 - 2x - 3$.
11. $a^5 - 41a - 120$ by $a^2 + 4a + 5$.
12. $25x^5 - x^3 - 8x - 2x^2$ by $5x^2 - 4x$.
13. $a^8 + a^6 + a^4 + a^2 + 3a - 1$ by $a + 1$.
14. $4y^4 - 9y^2 - 1 + 6y$ by $3y + 2y^2 - 1$.

15. $2a^4 - 5a^3b + 6a^2b^2 - 4ab^3 + b^4$ by $a^2 - ab + b^2$.

PROCESS | TEST

$$
\begin{array}{l|l}
2a^4 - 5a^3b + 6a^2b^2 - 4ab^3 + b^4 & a^2 - ab + b^2 \\
\underline{2a^4 - 2a^3b + 2a^2b^2} & \overline{2a^2 - 3ab + b^2} \\
\quad -3a^3b + 4a^2b^2 - 4ab^3 & \\
\quad \underline{-3a^3b + 3a^2b^2 - 3ab^3} & \\
\qquad\qquad a^2b^2 - ab^3 + b^4 & \\
\qquad\qquad \underline{a^2b^2 - ab^3 + b^4} &
\end{array}
\qquad
\begin{array}{l}
0 \div 1 \\
= 0
\end{array}
$$

NOTE. — It will be observed from the test that $0 \div 1 = 0$. In general, $0 \div a = 0$; that is, zero divided by any number equals zero (§ 542).

16. $6a^2 + 13ab + 6b^2$ by $3a + 2b$.

17. $3m^2 - 4am^3 + a^2m^4$ by $am - 1$.

18. $ax^3 - a^2x^2 - bx^2 + b^2$ by $ax - b$.

19. $20x^2y - 25x^3 - 18y^3 + 27xy^2$ by $6y - 5x$.

20. $a^4 - 4a^3x + 6a^2x^2 - 4ax^3 + x^4$ by $a^2 - 2ax + x^2$.

21.
$$
\begin{array}{l|l}
a^4 + 1 & a - 1 \\
\underline{a^4 - a^3} & \overline{a^3 + a^2 + a + 1 + \frac{2}{a-1}} \\
\quad a^3 + 1 & \\
\quad \underline{a^3 - a^2} & \\
\qquad a^2 + 1 & \\
\qquad \underline{a^2 - a} & \\
\qquad\quad a + 1 & \\
\qquad\quad \underline{a - 1} & \\
\qquad\qquad 2 &
\end{array}
$$

22.
$$
\begin{array}{l|l}
x^4 - 3x^3 + x^2 + 2x - 1 & x^2 - x - 2 \\
\underline{x^4 - x^3 - 2x^2} & \overline{x^2 - 2x + 1 + \frac{-x+1}{x^2 - x - 2}} \\
\quad -2x^3 + 3x^2 + 2x & \\
\quad \underline{-2x^3 + 2x^2 + 4x} & \\
\qquad\quad x^2 - 2x - 1 & \\
\qquad\quad \underline{x^2 - x - 2} & \\
\qquad\qquad -x + 1 &
\end{array}
$$

vide:

23. x^4+81 by $x-3$.
24. x^5+32 by $x+2$.
25. x^6-y^6 by x^2+y^2.
26. a^7+b^7 by $a+b$.
27. m^5-n^5 by $m+n$.
28. m^5+n^5 by $m+n$.
29. $a^6+5a^5-a^3+2a+3$ by $a-1$.
30. $x^7+2x^6-2x^4+2x^2-1$ by $x+1$.
31. $2x^8-x^7+2x^4-x^2+x^5+5$ by $x+1$.
32. $y^5+3y^4+5y^3+3y^2+3y+5$ by $y+1$.
33. $2n^5-4n^4-3n^3+7n^2-3n+2$ by $n-2$.
34. 1 by $1+x$ to five terms of the quotient.
35. 1 by $1-x$ to five terms of the quotient.
36. $a^3-6a^2+12a-8-b^3$ by $a-2-b$.
37. y^5+32x^5 by $16x^4+y^4-2xy^3-8x^3y+4x^2y^2$.
38. $x^3+y^3+z^3-3xyz$ by $x+y+z$.
39. $m^3+n^3+x^3+3m^2n+3mn^2$ by $m+n+x$.
40. $a^3-2a^2c+4ac^2-ax^2-4c^2x+2cx^2$ by $a-x$.
41. $a^3-b^3+c^3+3abc$ by $a^2+b^2+c^2+ab-ac+bc$.
42. $\frac{9}{16}m^4+\frac{4}{3}m-\frac{3}{4}m^3+\frac{16}{9}-\frac{7}{4}m^2$ by $\frac{3}{2}m^2-m-\frac{8}{3}$.
43. $\frac{1}{2}a^2x^2-\frac{3}{2}ax^3+\frac{9}{8}x^4-\frac{2}{9}a^4$ by $\frac{3}{4}x^2+\frac{1}{3}a^2-\frac{1}{2}ax$.
44. $\frac{1}{8}x^3+\frac{1}{27}y^3+z^3-\frac{1}{2}xyz$ by $\frac{1}{2}x+\frac{1}{3}y+z$.
45. $r^{2n}+11r^n+30$ by r^n+6.
46. $x^{3n-3}+y^{3n+3}$ by $x^{n-1}+y^{n+1}$.
47. x^n+y^n by $x+y$ to five terms of the quotient.
48. $2-3n^x+13n^{2x}+23n^{3x}-11n^{4x}+6n^{5x}$ by $2+3n^x$.
49. $a^{p+3}+a^{p+1}+a^{p-1}$ by $a^{p+1}+a^p+a^{p-1}$.
50. $x^{2r+1}y^{2s}+2x^{2r+3}y^{2s+1}+x^{2r+5}y^{2s+2}$ by $x^ry^{s-1}+x^{r+2}y^s$.
51. $6a^{2m}+5a^{2m-1}-10a^{2m-2}+20a^{2m-3}-16a^{2m-4}$ by $2a^m+$
${}^{-1}-4a^{m-2}$.

Divide, using detached coefficients:

52. x^5-5x+4 by x^2-2x+1.

PROCESS

$$\begin{array}{llll}
1+0+0+0-5+4 & & & \underline{|1-2+1} \\
\underline{1-2+1} & & & |1+2+3+4 \\
\quad\;\; 2-1+0 & & & =x^3+2x^2+3x+4 \\
\quad\;\; \underline{2-4+2} & & & \\
\qquad\quad 3-2-5 & & & \\
\qquad\quad \underline{3-6+3} & & & \\
\qquad\qquad\; 4-8+4 & & & \\
\qquad\qquad\; \underline{4-8+4} & & &
\end{array}$$

53. x^8+8x+7 by x^2+2x+1.

54. $a^6+38a+12$ by $a+2$.

55. $m^5-19m-6$ by $m+2$.

56. z^6-64 by $z-2$.

57. n^5+243 by $n+3$.

58. a^4-256 by $a+4$.

59. $a^6+27a^2-9a-10$ by $5-3a+a^2$.

60. $21x^4+4-8x^2+6x-29x^3$ by $3x-2$.

61. $16x^2-11x^3+2x^4+9-12x$ by $2x-3$.

62. $30x^4-36x+60x^2-62x^3+8$ by $5x-2$.

63. $x^7-2x^5-x^3-10x-36$ by $x-2$.

64. $y^4+7y-10y^2-y^3+15$ by y^2-2y-3.

65. $7x^3+2x^4-27x^2+16-8x$ by x^2+5x-4.

66. $28x^4+6x^3+6x^2-6x-2$ by $2+2x+4x^2$.

67. $25v^2-20v^3+3v^4+16v-6$ by $3v^2-8v+2$.

68. $4-18x+30x^2-23x^3+6x^4$ by $2x^2-5x+2$.

69. $32x^3+24x^4-25x-4-16x^2$ by $6x^2-x-4$.

70. $t^5-2t^4+\frac{1}{12}t^3+\frac{2}{5}t^2+\frac{1}{15}t+\frac{5}{4}$ by $t-\frac{3}{2}$.

71. $a^5-\frac{5}{4}a^4+\frac{29}{24}a^3-\frac{31}{24}a^2+\frac{5}{6}a-\frac{1}{4}$ by $a-\frac{3}{4}$.

72. $z^6-\frac{1}{6}z^5+\frac{5}{6}z^4-\frac{13}{36}z^3+\frac{37}{36}z^2-\frac{17}{36}z+\frac{7}{24}$ by $z^2-\frac{2}{3}z+\frac{1}{2}$.

SPECIAL CASES IN DIVISION

). By actual division,

1. $$\begin{cases} \dfrac{x^2-y^2}{x-y}=x+y. \\ \dfrac{x^3-y^3}{x-y}=x^2+xy+y^2. \\ \dfrac{x^4-y^4}{x-y}=x^3+x^2y+xy^2+y^3. \\ \dfrac{x^5-y^5}{x-y}=x^4+x^3y+x^2y^2+xy^3+y^4. \end{cases}$$

serve that the difference of the same powers of two num-
s exactly divisible by the difference of the numbers.

2. $$\begin{cases} \dfrac{x^2-y^2}{x+y}=x-y. \\ \dfrac{x^3-y^3}{x+y}=x^2-xy+y^2, \quad \text{rem., } -2y^3. \\ \dfrac{x^4-y^4}{x+y}=x^3-x^2y+xy^2-y^3. \\ \dfrac{x^5-y^5}{x+y}=x^4-x^3y+x^2y^2-xy^3+y^4, \quad \text{rem., } -2y^5. \end{cases}$$

serve that the difference of the same powers of two num-
s exactly divisible by the sum of the numbers only when
owers are even.

3. $$\begin{cases} \dfrac{x^2+y^2}{x-y}=x+y, \quad \text{rem., } 2y^2. \\ \dfrac{x^3+y^3}{x-y}=x^2+xy+y^2, \quad \text{rem., } 2y^3. \\ \dfrac{x^4+y^4}{x-y}=x^3+x^2y+xy^2+y^3, \quad \text{rem., } 2y^4. \end{cases}$$

erve that the sum of the same powers of two numbers is
actly divisible by the difference of the numbers.

$$4.\quad \begin{cases} \dfrac{x^2+y^2}{x+y}=x-y, \quad \text{rem.}, 2y^2. \\ \dfrac{x^3+y^3}{x+y}=x^2-xy+y^2. \\ \dfrac{x^4+y^4}{x+y}=x^3-x^2y+xy^2-y^3, \quad \text{rem.}, 2y^4. \\ \dfrac{x^5+y^5}{x+y}=x^4-x^3y+x^2y^2-xy^3+y^4. \end{cases}$$

Observe that the sum of the same powers of two numbers is exactly divisible by the sum of the numbers only when the powers are odd.

134. Hence, when n is a positive integer,

PRINCIPLES. — 1. *x^n-y^n is always divisible by $x-y$.*

2. *x^n-y^n is divisible by $x+y$ only when n is even.*

3. *x^n+y^n is never divisible by $x-y$.*

4. *x^n+y^n is divisible by $x+y$ only when n is odd.*

"Divisible" means "exactly divisible."

135. The following **law of signs** may be inferred readily:

When $x-y$ is the divisor, the signs in the quotient are plus.

When $x+y$ is the divisor, the signs in the quotient are alternately plus and minus.

136. The following **law of exponents** also may be inferred:

The quotient is homogeneous, the exponent of x decreasing and that of y increasing by 1 in each successive term.

EXERCISES

137. Find quotients by inspection:

1. $\dfrac{a^3-b^3}{a-b}$.
2. $\dfrac{m^3-n^3}{m-n}$.
3. $\dfrac{a^3-8}{a-2}$.
4. $\dfrac{a^3+b^3}{a+b}$.
5. $\dfrac{m^3+n^3}{m+n}$.
6. $\dfrac{c^3+27}{c+3}$.

ınd quotients by inspection:

'. $\frac{a^3-125}{a-5}$. 10. $\frac{r^7-s^7}{r-s}$. 13. $\frac{1+a^5}{1+a}$.

ı. $\frac{n^3+64}{n+4}$. 11. $\frac{n^3-1}{n-1}$. 14. $\frac{x^5-32}{x-2}$.

ı. $\frac{m^5+n^5}{m+n}$. 12. $\frac{x^6-1}{x+1}$. 15. $\frac{a^7+128}{a+2}$.

. Find five exact binomial divisors of a^6-x^6.

SOLUTION

$-x^6$ is divisible by $a-x$ (Prin. 1).
$-x^6$ is divisible by $a+x$ (Prin. 2).
nce $a^6-x^6=(a^2)^3-(x^2)^3$, a^6-x^6 may be regarded as the differ- of two cubes, and is, therefore, divisible by a^2-x^2 (Prin. 1).
ıce $a^6-x^6=(a^3)^2-(x^3)^2$, a^6-x^6 may be regarded as the difference o squares, and is, therefore, divisible by a^3-x^3 (Prin. 1).
ıce $a^6-x^5=(a^3)^2-(x^3)^2$, a^6-x^6 may be regarded as the differ- of two squares, and is, therefore, divisible by a^3+x^3 (Prin. 2).
ıerefore, the exact binomial divisors of a^6-x^6 are $a-x$, $a+x$, x^2, a^3-x^3, and a^3+x^3.

. Find an exact binomial divisor of a^6+x^6.

SOLUTION

nce $a^6+x^6=(a^2)^3+(x^2)^3$, a^6+x^6 may be regarded as the sum of ubes of a^2 and x^2, and is, therefore, divisible by a^2+x^2 (Prin. 4).

ınd exact binomial divisors:

ı. a^2-m^2. 24. x^7+a^7. 30. a^4-b^4, four.

ı. a^3-m^3. 25. $a^{10}+b^{10}$. 31. a^6-1, five.

ı. b^3+x^3. 26. $a^{10}+b^5$. 32. a^8-b^8, six.

.. x^5-a^5. 27. $a^{12}+b^{12}$. 33. $a^{10}-b^{10}$, five.

ı. c^5+n^5. 28. a^3-27. 34. $a^{16}-b^{16}$, eight.

. a^6+b^6. 29. a^6-27. 35. $a^{12}-b^{12}$, nine.

Algebraic Representation

138. 1. Express m dollars in terms of cents; m cents in terms of dollars.

2. Find the value of x that will make $6x$ equal to 48.

3. By what number must 25 be multiplied to produce 300? 10 to produce x? r to produce s?

4. Represent (the third power of a minus the fifth power of x) divided by (m plus n^2).

5. Express the multiplicand when lmn is the product and lm the multiplier.

6. Find an expression for 5 per cent of x; y per cent of z.

7. It takes a men c days to do a piece of work. How long will it take one man to do it? 2 men? x men?

8. At a factory where N persons were employed, the weekly pay roll was P dollars. Find the average earnings of each person per week.

9. A train ran M miles in H hours and m miles in the succeeding h hours. Find its average rate per hour during each period and during the whole time.

10. A farmer has hay enough to last m cows for n days. How long will it last $(a-b)$ cows?

11. Indicate the quotient of $m+n$ divided by the number whose first digit is x, second digit y, and third digit z.

12. A dealer bought n 50-gallon barrels of paint at c cents per gallon. He sold the paint and gained g dollars. Find the selling price per gallon.

13. If it takes b men c days to dig part of a well, and d men e days to finish it, how long will it take one man to dig the well alone?

Equations and Problems

139. 1. Find the value of x in the equation $bx - b^2 = cx - c^2$.

SOLUTION

$$bx - b^2 = cx - c^2.$$

Transposing, $bx - cx = b^2 - c^2$.

Collecting coefficients of x, $(b - c)x = b^2 - c^2$.

Dividing by $b - c$, $x = \dfrac{b^2 - c^2}{b - c} = b + c.$

2. Find the value of x in the equation $x - a^3 = 2 - ax$.

SOLUTION

$$x - a^3 = 2 - ax.$$

Transposing, $ax + x = a^3 + 2$.

Collecting coefficients of x, $(a + 1)x = a^3 + 2$.

Dividing by $a + 1$, $x = \dfrac{a^3 + 2}{a + 1} = a^2 - a + 1 + \dfrac{1}{a + 1}.$

Find the value of x in:

3. $cx - c^3 - d^3 + dx = 0.$

4. $a^2 - ax - 2ab + bx + b^2 = 0.$

5. $2n^2 + 5n + x = n^3 - nx - 2.$

6. $7a - 10 = a^2 - ax + 5x.$

7. $x - 1 - c = cx - c^3 - c^4.$

8. $2m^3 - mx + nx - 2n^3 = 0.$

9. $3ab - a^2 - 2bx = 2b^2 - ax.$

10. $a^2x - a^3 + 2a^2 + 5x - 5a + 10 = 0.$

11. $ax - 2bx + 3cx = ab - 2b^2 + 3bc.$

12. $cx - c^4 - 2c^3 - 2c^2 = 2c - x + 1.$

13. $9a^2 + 4mx = -(3ax - 16m^2).$

14. $x + 6n^4 - 4n^3 = 1 - 3nx + 2n - n^2.$

15. $n^2x - 3m^2n^3 + nx + 3m^2 + x = 0.$

16. $x - 3b^2 - 192b^2c^3 - 4cx + 16c^2x = 0.$

Solve the following problems and verify the solutions:

17. George and Henry together had 46 cents. If George had 4 cents more than half as many as Henry, how many cents had each?

SOLUTION

Let $x =$ the number of cents George had.

Then, $x - 4 =$ the number of cents George had less 4,

and $2(x - 4) =$ the number of cents Henry had;

$\therefore x + 2(x - 4) = 46.$

Solving, $x = 18$, the number of cents George had,

and $2(x - 4) = 28$, the number of cents Henry had.

VERIFICATION

The answers obtained should be tested by the conditions of the problem. If they satisfy the conditions of the problem, the solution is presumably correct.

1st condition: They had together 46 cents.

$$18 + 28 = 46.$$

2d condition: George had 4 cents more than half as many as Henry.

$$18 = \tfrac{1}{2} \text{ of } 28 + 4.$$

18. In a certain election at which 8000 votes were polled, B received 500 votes more than half as many as A. How many votes did each receive?

19. A had \$40 more than B; B had \$10 more than one third as much as A. How much money had each?

20. In 2 years A will be twice as old as he was 2 years ago. How old is he?

21. Two wheelmen start at the same time from A to ride to B. One rides at the rate of 10 miles an hour, and rests 3 hours; the other rides at the rate of 8 miles an hour, and by resting only 1 hour arrives at B as soon as the faster rider. *How many* hours are occupied in making the trip? How far *is it from A to B?*

22. It cost a man 60¢ to send a telegram at "30–2", that is, 30¢ for the first 10 words and 2¢ for each additional word. How many words did the message contain?

SOLUTION

Let x be the number of words in the message.

Then, $x - 10$ will represent the number of words in excess of 10 words.

$$\therefore 30 + 2(x - 10) = 60.$$

Solving, $x = 25$, the number of words.

VERIFICATION

30¢ for 10 words + 30¢ for 15 additional words = 60¢.

23. How many words can be sent by telegraph from New Haven to New York for 75¢ at the day rate, "25–2"?

24. A long-distance telephone message cost me $1.25. The rate was 50¢ for the first 3 minutes and 15¢ for each additional minute. How long did the conversation last?

25. The day rate for a telegram between New Orleans and New York is "60–4" and the night rate is "40–3." A message of a certain number of words cost 25¢ less to send at night than in the daytime. Find the number of words.

26. Separate 24 into two parts, x and $24 - x$, such that one part shall be 3 less than twice the other.

27. Separate 52 into two parts such that 2 times one part shall be 4 greater than 3 times the other.

28. Mary bought 17 apples for 61 cents. For a certain number of them she paid 5 cents each, and for the rest she paid 3 cents each. How many of each kind did she buy?

29. George is $\frac{1}{2}$ as old as his father; a years ago he was $\frac{1}{3}$ as old as his father. What is the age of each?

30. A rug 3 feet longer than it is wide, placed on the floor of a certain room, leaves a margin of 2 feet on every side. If the area of the floor is 172 square feet greater than the area of the rug, what are the dimensions of the floor?

REVIEW

140. 1. Define a term; similar terms; the degree of a term; the degree of an expression.

Illustrate symmetrical expression; homogeneous expression.

Simplify:

2. $x^3 + 2a\sqrt{xy} - 3mn - 4x^3 - 5a\sqrt{xy} + 3x^3 + 4a\sqrt{xy} + 4mn.$

3. $5x^3 + 3x^2y + 4xy^2 - y^3 - \sqrt{x} + 6y^3 + xy^2 - \sqrt{y} - 5x^2y - 7x^3 - 5xy^2 + x^3 + 2\sqrt{x} + x^2y - 6y^3 + \sqrt{y} - \sqrt{x} - 2x^2y + 3xy^2 + x^3.$

4. How may a parenthesis preceded by a minus sign be removed from an algebraic expression without changing the value of the expression?

Simplify:

5. $x^5 - (x^5 - 5x^4y + 10x^3y^2 - 10x^2y^3 + 5xy^4 - y^5).$

6. $\frac{1}{2}a - \frac{5}{6}x - (\frac{3}{4}a - \frac{1}{2}x) - (3b - \frac{11}{4}x - \frac{2}{3}a) + \frac{1}{6}a.$

7. $x^2 - (2xy - y^2) - (x^2 + xy - y^2) - x^2 - \overline{2xy - y^2} + 5y^2.$

8. $m + 2\{2m - [n + 3p - (4p - 3n) - 5n + 2m] - 7p\}.$

9. What are the various ways of indicating multiplication in algebra?

Expand:

10. $(m - x)(m + x).$

11. $(x^2 + 4)(x^2 - 3).$

12. $(x^3 + x^2)(x + 1).$

13. $(x - 1)(1 + x).$

14. $(a^m + b^n)(a^n - b^m).$

15. $(a + b + c)(a + b - c).$

16. $(x + y + z)(x - y + z).$

17. $(m + n - p)(m - n + p).$

18. Why should the terms of the dividend and divisor usually be arranged, before division, according to the ascending or the descending powers of some letter?

19. What is the advantage of using detached coefficients?

Arrange terms and divide, using detached coefficients:

20. $x^5 - x + 2x^3 - 8 - 2x^4 + 12x^2$ by $x + 1$.

21. $x^4 - 4x + 5x^2 - 4x^3 + 1$ by $1 - 3x + x^2$.

22. $a^7 - 12a^2 - a + 12$ by $a^3 - 3 + 4a - 2a^2$.

Simplify:

23. $1 - \{1 - [x^2 - 3 - (2x - 4)^2 + 3x^2 + 1] - (x - 4)^2\} - 1$.

24. $x - \{5x - [6x - (7x - \overline{8x - 9x}) - 10x] + 11x\} + 9x$.

25. $1 - \{-[-(1 - x) - 1] - 1\} - \{x - (5 - 3x) - 7 + x\}$.

Collect, in order, the coefficients of x, y, and z:

26. $ax + ay + az - bx - by - bz$.

27. $ax - 2y + cz + by - 12x + 4z$.

28. $3mx - nx + by - y + 3cz - 4z$.

29. $py - y - 4z + bz - x + mx - nx - z$.

30. $cx - by - 3az + x - y - 4z + z - y$.

31. State the law of signs for multiplication; for division.

32. What is the sign of the product of an even number of negative factors? of an odd number of negative factors?

Expand:

33. $(a - b)(a + b)(a^2 + b^2)$.

34. $(1 - x)(1 + x)(1 + x^2)(1 + x^4)$.

35. $(1 - x)(1 + x)(1 - x)(1 + x)$.

36. $(a^3 + 3a^2y + 3ay^2)(a^2 - 2ay + y^2)$.

37. $(x^{2n} + 2x^ny^n + y^{2n})(x^{2n} - 2x^ny^n + y^{2n})$.

38. $(\frac{1}{4}x^2 + \frac{1}{3}xy + \frac{1}{9}y^2)(\frac{1}{4}x^2 - \frac{1}{3}xy + \frac{1}{9}y^2)$.

39. $(.2a^2 - .8a + 1.6)(.1a^2 + .4a + .8)$.

40. Give a rule for multiplying a monomial by a monomial; for dividing a polynomial by a polynomial.

41. State and illustrate two ways of testing the correctness of a result in algebraic multiplication; in algebraic division.

Expand, using detached coefficients; test results:

42. $(a^4+a^3+a^2+a+1)(a-1)$.

43. $(x^5-x^4+x^3-x^2+x-1)(x+1)$.

44. $(a^5+2a^4+4a^3+8a^2+16a+32)(a-2)$.

Divide, and test results:

45. $4-10b^2-5b+b^8$ by $3b-2b^2+b^3-1$.

46. $m^{10}-6m^3+5m-2$ by $2m^3-2+m^4-3m$.

47. $127a^3-20a+a^7-100a^2+16-160a^4$ by a^3-6a^2+5a-4.

48. $b^{10}+29b^4-22-61b^2+210b-170b^3$ by $b^4-5b+2b^2-11$.

Simplify:

49. $a-(2b+5a)(6b-3a)-2b-6[3a^2-4ab-2b^2]$.

50. $x-\{3y+[4x-2(y+3x)-3y]^2-(5y+2x)^2-8y\}$.

51. $(a^2+ab-b^2)^2-(a^2-ab-b^2)^2-4ab(a^2-b^2)$.

52. $a^2-[-b^2+b(5b-3a)-\{-\overline{3ab+a^2}-b(a-\overline{2a+2b})\}]$.

Square:

53. $2x-3y$.

54. x^4-ax^2.

55. $5a^2-1$.

56. $10-3x$.

57. n^x-m^y.

58. $7x-b^2y$.

59. $a+b-c+d$.

60. $2a-3b-4c$.

61. $x^{n-1}-y-a^2$.

62. What laws are illustrated by $a(bc)=b(ac)$?

63. Show that $a^5\cdot a^6=a^{11}$; that $a^{11}\div a^6=a^5$.

64. In what respect do $(a-b)$ and $(b-a)$ differ? Expand and compare $(a-b)^2$ and $(b-a)^2$.

Collect, in order, the coefficients of x, y, and z:

65. $16ny - 16mx + ax + by + cx - 2y$.

66. $mx + ny + az + 2ax - 2my + 2nz$.

67. $x - y - az + 8mx + aby - x^2 + y^2 + z$.

68. $a^2x + b^2y - 2ax - 2cz + c^2z + x + y + z$.

69. $m^2x - n^2y + m^2y - n^2x - 2mnx - 2mny - n^2z + z$.

70. $4(ax - by + cz) - 2(bx - ay - dz) - 2(x - y + z)$.

71. Show why a broader definition is necessary for multiplication in algebra than in arithmetic.

Expand:

72. $(5a - 4y)(5a - 3y)$.

73. $(6x - 4y)(3x + 5y)$.

74. $(3x + ay)(3x + by)$.

75. $(2a^2x - 5b^2y)(4a^2x - 3b^2y)$.

76. $(6amn + 5p)(6amn - 3p)$.

77. $(3a^{n+1} - 2b^{n-1})(2a^{n+1} - 3b^{n-1})$.

78. $(x + y)(x - y)(x^2 + y^2)(x^4 + y^4)(x^8 + y^8)$.

79. $(m^8 + 1)(m^4 + 1)(m^2 + 1)(m + 1)(m - 1)$.

80. $(16x^4 + 1)(4x^2 + 1)(2x + 1)(2x - 1)$.

81. For what values of n is $x^n + y^n$ divisible by $x + y$? by $x - y$? When is $x^n - y^n$ divisible by $x + y$? by $x - y$?

82. State the law of signs for the quotient when $x^n + y^n$ or $x^n - y^n$ is divided by $x + y$ or $x - y$; the law of exponents.

Divide:

83. $x^{2a+2} - x^{a+1}y^a - 2y^{2a} + 3y^az^{a-1} - z^{2a-2}$ by $x^{a+1} + y^a - z^{a-1}$.

84. $6a^7 + \frac{39}{16}a^2y^5 - \frac{33}{32}ay^6 + \frac{21}{64}y^7$ by $a^3 + \frac{1}{2}a^2y - \frac{1}{4}ay^2 + \frac{1}{8}y^3$.

85. $a^2c - ab^2 + acd - ad^2 - abc + b^3 - bcd + bd^2 - ac^2 + cb^2 - c^2d + cd^2$ by $ac - b^2 + cd - d^2$.

FACTORING

141. An expression is **rational**, if in its simplest form it contains no root sign of any kind; and **integral**, if in its simplest form it contains no literal number in any denominator or divisor.

$x^3 - 6x^2 + 11x - 6$ and $\frac{1}{2}x - 3$ are rational integral expressions; $\frac{x}{a} + b$ is rational but not integral; $a + \sqrt{x}$ is integral but not rational.

Until noted farther on, the term factor (§ 24) will be understood to mean *rational integral factor.*

142. A number that has no factors except itself and 1 is called a **prime number.**

143. The process of separating a number into its factors is called **factoring.** Usually the *prime* factors are sought.

144. To factor a monomial.

EXERCISES

1. In each of the following, if xy is one factor, find the other: $6x^5y$, $15x^4y^2$, $2x^3y^3$, $a^2x^2b^2y^2$, $-mnxy$, $-xy$.

2. In each of the following, if abc is one factor, find the other: a^2bc, ab^2c, abc^2, $-a^2b^2c^2$, $-a^2bc$, $-\frac{1}{3}abc$.

3. Find two equal positive factors of x^2; of $9a^2x^2$; of $64m^4$.

4. Find two equal negative factors of $25x^2$; of $16a^2$; of $9a^6$.

145. A factor of two or more numbers is called a **common factor** of them

146. To factor a polynomial whose terms have a common factor.

EXERCISES

1. What are the factors of $3a^2xy - 6ax^2y + 9axy^2$?

PROCESS

$$3a^2xy - 6ax^2y + 9axy^2$$
$$= 3axy(a - 2x + 3y)$$

EXPLANATION. — By examining the terms of the polynomial, it is seen that $3axy$ is a factor of every term. Dividing by this common factor gives the other factor.

Hence, the factors are $3axy$, the monomial factor, and $(a - 2x + 3y)$, the polynomial factor.

TEST. — The product of the factors should equal the given expression; thus, $3axy(a - 2x + 3y) = 3a^2xy - 6ax^2y + 9axy^2$.

Factor, and test each result:

2. $5x^3 - 5x^2$.
3. $8x^2 + 2x^4$.
4. $3x^3 - 6x^2y$.
5. $4a^2 - 6ab$.
6. $5m^2 - 3mn$.
7. $3x^3y^2 - 3x^2y^3$.
8. $5m^4n - 10m^3n^2$.
9. $4a^3b - 6a^2b^2$.
10. $5x^4 - 10x^3 - 5x^2$.
11. $3a^4 - 2a^3b + a^2b^2$.
12. $x^{12} + x^{11} + x^{10} - x^9$.
13. $3m^5 - 12m^3n^2 + 6mn^4$.
14. $ac - bc - cy - abc$.
15. $3x^3y^3 - 3x^2y^2 + 12xy$.
16. $16a^2b^3c^4 - 24a^3b^2c^3 + 32a^3b^4c^3$.
17. $60m^2n^3r^2 - 45m^3n^2r^3 + 90m^4n^3r^2$.
18. $12a^2by - 18ab^2y^2 + 24a^3b^2y^2$.
19. $14a^2mn^2 - 21a^3m^2n^3 - 49a^4mn^2$.
20. $12x^2y^2z^3 - 16x^2y^2z^2 - 20x^3y^3z^3$.
21. $25c^2dx^3 + 35c^3d^2x^4 - 55c^2d^2x^5$.

147. To factor a polynomial whose terms may be grouped to show a common polynomial factor.

EXERCISES

1. Factor $ax + ay + bx + by$.

SOLUTION

$$ax + ay + bx + by = a(x + y) + b(x + y)$$
$$= (a + b)(x + y).$$

2. Factor $ax - ay - bx + by$.

SOLUTION

$$ax - ay - bx + by = a(x - y) - b(x - y)$$
$$= (a - b)(x - y).$$

Observe that, when the first two terms are factored, $(x - y)$ is found to be the binomial factor. Since $(x-y)$ is to be a factor of the other two terms, the monomial factor is $-b$, not $+b$, for $(-bx+by) \div (x-y) = -b$.

3. Factor $cx + y - dy + cy - dx + x$.

SOLUTION

$$cx + y - dy + cy - dx + x$$

Arranging terms, $= cx - dx + x + cy - dy + y$
$$= (c - d + 1)x + (c - d + 1)y$$
$$= (c - d + 1)(x + y).$$

Factor, and test each result, especially for signs:

4. $am - an + mx - nx$.
5. $bc - bd + cx - dx$.
6. $pq - px - rq + rx$.
7. $ay - by - ab + b^2$.
8. $x^2 - xy - 5x + 5y$.
9. $b^2 - bc + ab - ac$.
10. $x^2 + xy - ax - ay$.
11. $c^2 - 4c + ac - 4a$.
12. $2x - y + 4x^2 - 2xy$.
13. $1 - m + n - mn$.
14. $2p + q + 6p^2 + 3pq$.
15. $ar - rs - ab + bs$.
16. $x^3 + x^2 + x + 1$.
17. $y^3 + y^2 - 3y - 3$.
18. $x^5 + x^3 + x^2y + y$.
19. $2 - 2n - n^2 + n^3$.
20. $x^2 - x - a + ax$.
21. $3x^3 - 15x + 10y - 2x^2y$.
22. $12a^3 - 8ab - 3a^4 + 2a^2b$.
23. $3m^2n - 9mn^2 + am - 3an$.
24. $15ab^2 - 9b^2c - 35ab + 21bc$.
25. $16ax + 12ay - 8bx - 6by$.
26. $ax^2 - ax - axy + ay + x - 1$.
27. $xy + x - 3y^2 - 3y - 4y - 4$.
28. $ax - a - bx + b - cx + c$.
29. $mx - nx - x - my + ny + y$.
30. $bx^2 - b - xy - y + yx^2 - bx$.
31. $m^2 + mn + mn + n^2 + m + n$

148. To factor a trinomial that is a perfect square.

Since by multiplication, §§ 105, 108,

$(a+b)(a+b)=a^2+2ab+b^2$ and $(a-b)(a-b)=a^2-2ab+b^2$,

$a^2+2ab+b^2=(a+b)(a+b)$ and $a^2-2ab+b^2=(a-b)(a-b)$.

These two trinomials are **perfect squares**, for each may be separated into two *equal* factors. They are types, showing the form of all **trinomial squares**, for a and b may represent any two numbers.

149. A trinomial is a perfect square, therefore, if these two conditions are fulfilled:

1. Two terms, as $+a^2$ and $+b^2$, must be perfect squares.
2. The other term must be numerically equal to twice the product of the square roots of the terms that are squares.

$25x^2-20xy+4y^2$ is a perfect square, for $25x^2=(5x)^2$, $4y^2=(2y)^2$, and $-20xy=-2(5x)(2y)$.

150. Every number has two square roots, one positive and the other negative. In factoring, usually only the *positive* square root is taken.

Thus, $\sqrt{25}=5$ or -5, for $5\cdot 5=25$ and $(-5)(-5)=25$.

$a^2+2ab+b^2=(a+b)(a+b)$ or $(-a-b)(-a-b)$, but we usually factor trinomial squares in the first way only.

RULE. — *Connect the square roots of the terms that are squares with the sign of the other term, and indicate that the result is to be taken twice as a factor.*

From any expression that is to be factored, the *monomial* factors should usually first be removed.

Thus, $2a^3-4a^2+2a=2a(a^2-2a+1)=2a(a-1)^2$.

EXERCISES

151. Factor, and test each result:

1. $x^2+2xy+y^2$.
2. $p^2-2pq+q^2$.
3. $c^2+2cd+d^2$.
4. $m^2-2mn+n^2$.
5. x^2-2x+1.
6. x^2+4x+4.

7. x^2+6x+9.
8. $4-4a+a^2$.
9. $4a-4a^2+a^3$.
10. $m^2-8m+16$.
11. $a^2-16a+64$.
12. $5x^2+30x+45$.
13. $3x^2+6xy+3y^2$.
14. $2m^2-4mn+2n^2$.
15. $1+4b+4b^2$.
16. $1-6a^3+9a^6$.
17. $10x^2-20x+10$.
18. $16p^2-24p+9$.
19. $9x^2-42x+49$.
20. $9+42b^3+49b^6$.
21. $9m^8-6m^4+1$.
22. $4x^2y^2-20xy+25$.
23. $4x^2+12xyz+9y^2z^2$
24. $9a^2m^2-6am+1$.
25. $2x+20a^2x+50a^4x$.
26. $18a^2b+60ab^2+50$
27. $a^2x^6-2ax^3by^3+b^2y^6$
28. $x^{2n}-2x^ny^nz^n+y^{2n}z^{2n}$

When either or both of the squares are squares of] nomials, the expression may be factored in a similar mann

29. Factor $x^2+6x(x-y)+9(x-y)^2$.

SOLUTION

$$\begin{aligned}
&x^2+6x(x-y)+9(x-y)^2\\
&=[x+3(x-y)][x+3(x-y)]\\
&=(x+3x-3y)(x+3x-3y)\\
&=(4x-3y)(4x-3y).
\end{aligned}$$

30. Factor $(a-b)^2+2(a-b)(b-c)+(b-c)^2$.

SOLUTION

$$\begin{aligned}
&(a-b)^2+2(a-b)(b-c)+(b-c)^2\\
&=[(a-b)+(b-c)][(a-b)+(b-c)]\\
&=(a-b+b-c)(a-b+b-c)\\
&=(a-c)(a-c).
\end{aligned}$$

TEST. — When $a=3$, $b=2$, and $c=1$,

$$(a-b)^2+2(a-b)(b-c)+(b-c)^2=1^2+2\cdot1\cdot1+1^2=4,$$

and

$$(a-c)(a-c)=2\cdot2=4.$$

Factor, and test each result:

31. $x^2+2x(x-y)+(x-y)^2$.
32. $t^2-4t(t-1)+4(t-1)^2$.
33. $(r+s)^2-4(r+s)+4$
34. $c^2-6c(a-c)+9(a$

5. $16-24(t-l)+9(t-l)^2$.

6. $14(x-y)+(x-y)^2+49$.

7. $(a+b)^2-2(a+b)(b+c)+(b+c)^2$.

8. $(a-2x)^2+4(a-2x)(2x-b)+4(2x-b)^2$.

9. $16(a-x)^2+32(a-x)(x+b)+16(x+b)^2$.

0. $(a+3b)^2-4(a+3b)(3b-2c)+4(3b-2c)^2$.

1. $(x^2+x+1)^2+2(x+1)(x^2+x+1)+(x+1)^2$.

2. $(a+b+c)^2+2(a+b-c)(a+b+c)+(a+b-c)^2$.

To factor the difference of two squares.

ultiplication, $(a+b)(a-b)=a^2-b^2$.

fore, $a^2-b^2=(a+b)(a-b)$.

. — *Find the square roots of the two terms, and make n one factor and their difference the other.*

imes the factors of a number may themselves be factored.

EXERCISES

1. Factor b^2-y^2.

ION. $b^2-y^2=(b+y)(b-y)$.

actor x^2-1.

ION. $x^2-1=(x+1)(x-1)$.

actor x^4-1.

ION. $x^4-1=(x^2+1)(x^2-1)$
$=(x^2+1)(x+1)(x-1)$.

ve into their simplest factors:

$-m^2$.	10. x^4-81.	16. $25x^2-1$.
$-y^2$.	11. a^4-b^4.	17. $144m^4-1$.
-16.	12. $a^{16}-b^8$.	18. $36a^4-225$.
-9.	13. $9a^2-49b^2$.	19. $121b^2-a^2c^2$.
$-c^2$.	14. $a^2x^2-4c^2$.	20. $100a^2-81y^2$.
-49.	15. m^4-16n^4.	21. $64x^2-625y^2$.

22. $169 - a^2c^2$.
23. $400\,x^2 - 36\,y^2$.
24. $144\,m^2 - 16\,n^2$.
25. $x^4y^4 - 256$.
26. $2\,a^8 - 2\,b^8$.
27. $4\,m^4 - 4\,b^4$.
28. $3\,x^4 - 3\,y^4$.
29. $5\,x^8 - 5$.
30. $3\,a^5 - 3\,a$.
31. $x^3 - xy^2$.
32. $x^2 - .01$.
33. $a^4 - \frac{1}{16}$.
34. $x^{2n} - y^{2n}$.
35. $x^{2n-2} - y^{4m}$.
36. $x^{2n+1} - xy^{2n}$.

When either or both of the squares are squares of polynomials, the expression may be factored in a similar manner.

37. Factor $25\,a^2 - (3\,a + 2\,b)^2$.

Solution

One factor is $5\,a + (3\,a + 2\,b)$ and the other is $5\,a - (3\,a + 2\,b)$.

$$5\,a + (3\,a + 2\,b) = 5\,a + 3\,a + 2\,b = 8\,a + 2\,b = 2(4\,a + b).$$

$$5\,a - (3\,a + 2\,b) = 5\,a - 3\,a - 2\,b = 2\,a - 2\,b = 2(a - b).$$

$$\therefore 25\,a^2 - (3\,a + 2\,b)^2 = 2(4\,a + b)2(a - b)$$
$$= 4(4\,a + b)(a - b).$$

Factor:

38. $a^2 - (b + c)^2$.
39. $b^2 - (2\,a + b)^2$.
40. $a^2 - (a + b)^2$.
41. $4\,c^2 - (b + c)^2$.
42. $9\,b^2 - (a - x)^2$.
43. $9\,a^2 - (2\,a - 5)^2$.
44. $x^4 - (3\,x^2 - 2\,y)^2$.
45. $49\,a^2 - (5\,a - 4\,b)^2$.

46. Factor $(3\,a - 2\,b)^2 - (2\,a - 5\,b)^2$.

Solution

$$(3\,a - 2\,b)^2 - (2\,a - 5\,b)^2$$
$$= [(3\,a - 2\,b) + (2\,a - 5\,b)][(3\,a - 2\,b) - (2\,a - 5\,b)]$$
$$= (3\,a - 2\,b + 2\,a - 5\,b)(3\,a - 2\,b - 2\,a + 5\,b)$$
$$= (5\,a - 7\,b)(a + 3\,b).$$

Factor:

47. $(2\,a + 3\,b)^2 - (a + b)^2$.
48. $(5\,a - 3\,b)^2 - (a - b)^2$.
49. $(2\,x + 5)^2 - (5 - 3\,x)^2$.
50. $(a - 2\,b)^2 - (a - 5)^2$.

. $(2x-3y)^2-(3y+z)^2$.

54. $(9x+6y)^2-(4x-3y)^2$.

. $(5b-4c)^2-(3a-2c)^2$.

55. $(x^3+x^2)^2-(2x+2)^2$.

. $(4x-3y)^2-(2x-3a)^2$.

56. $(a+b+c)^2-(a-b-c)^2$.

'. Factor a^2+4-c^2-4a.

Solution

$$a^2+4-c^2-4a$$

rranging terms,
$$\begin{aligned} &= (a^2-4a+4)-c^2 \\ &= (a-2)^2-c^2 \\ &= (a-2+c)(a-2-c). \end{aligned}$$

3. Factor $a^2+b^2-c^2-4-2ab+4c$.

Solution

$$a^2+b^2-c^2-4-2ab+4c$$

rranging terms,
$$\begin{aligned} &= a^2-2ab+b^2-c^2+4c-4 \\ &= (a^2-2ab+b^2)-(c^2-4c+4) \\ &= (a-b)^2-(c-2)^2 \\ &= (a-b+c-2)(a-b-c+2). \end{aligned}$$

ctor, and test each result:

. $a^2-2ax+x^2-n^2$.

65. $b^2-x^2-y^2+2xy$.

. $b^2+2by+y^2-n^2$.

66. $4c^2-x^2-y^2-2xy$.

. $1-4q+4q^2-a^2$.

67. $9c^2-x^2-y^2+2xy$.

. $r^2-2rx+x^2-16t^2$.

68. $x^3-a^2x-4b^2x-4abx$.

. $9a^2b-6ab^2+b^3-4bc^2$.

69. $bc^2-9a^2b-b^3-6ab^2$.

. $c^2-a^2-b^2-2ab$.

70. $ab^2-4a^3-12a^2c-9ac^2$.

71. $a^2-2ab+b^2-c^2+2cd-d^2$.

72. $x^2-2xy+y^2-m^2+10m-25$.

73. $4x^2+9-12x+10mn-m^2-25n^2$.

74. $x^2-a^2+y^2-b^2+2xy-2ab$.

154. To factor a trinomial of the form $x^2 + px + q$.

By multiplication,

$$(x + a)(x + b) = x^2 + (a + b)x + ab.$$

This trinomial consists of x^2, an x-term, and an a
term; and therefore has the type form $x^2 + px + q$.

Therefore, by reversing the process of multiplicatio
nomial of this form may be factored by *finding two fa
q (the absolute term) such that their sum is p (the coeffici
and adding each factor of q to x.*

Thus,
$$x^2 + 8x + 15 = (x + 3)(x + 5),$$
$$x^2 - 8x + 15 = (x - 3)(x - 5),$$
$$x^2 + 2x - 15 = (x - 3)(x + 5),$$
$$x^2 - 2x - 15 = (x + 3)(x - 5).$$

EXERCISES

155. 1. Resolve $x^2 - 13x - 48$ into two binomial fac

SOLUTION. — The first term of each factor is evidently x.

Since the product of the second terms of the two binomial
-48, the second terms must have opposite signs; and since tl
braic sum, -13, is negative, the negative term must be nu
larger than the positive term.

The two factors of -48 whose sum is negative may be 1 a
2 and -24, 3 and -16, 4 and -12, or 6 and -8. Since the
sum of 3 and -16 is -13, 3 and -16 are the factors of -48 so

$$\therefore x^2 - 13x - 48 = (x + 3)(x - 16).$$

2. Factor $72 - m^2 - m$.

SOLUTION. — Arranging the trinomial according to the de
powers of m,

$$72 - m^2 - m = -m^2 - m + 72$$

Making m^2 positive,

$$= -(m^2 + m - 72)$$
$$= -(m - 8)(m + 9)$$
$$= (-m + 8)(m + 9)$$
$$= (8 - m)(9 + m).$$

Separate into simplest factors and test each result by assign-
a numerical value to each letter:

3. $x^2+7x+12$.	16. $x^2+5ax+6a^2$.
4. $y^2-7y+12$.	17. $x^2-6ax+5a^2$.
5. $p^2-8p+12$.	18. $y^2-4by-12b^2$.
$r^2+8r+12$.	19. $y^2-3ny-28n^2$.
$15+2a-a^2$.	20. $z^2-anz-2a^2n^2$.
b^2+b-12.	21. $x^4+19cx^2+90c^2$.
$30-r^2+r$.	22. $x^6+12ax^3+20a^2$.
c^2-c-72.	23. $x^{10}-11b^2x^5+24b^4$.
$c^2-5c-14$.	24. $5nx^2-55nx+150n$.
$x^2-x-110$.	25. $3a^2bx^2-3a^2bx-6a^2b$.
$-a^2-9a+52$.	26. $4ax+2ax^2-48a$.
$a^2+8a-128$.	27. $11a^2x-55ax+66x$.
5. $-x^2+25x-100$.	28. $20bx+10b^2-630x^2$.

29. Factor $x^2-(c+d)x+cd$.

SUGGESTION. — Write the trinomial in the standard form,

$$x^2+(-c-d)x+(-c)(-d).$$

0. Factor $x^2-(a-d)x-ad$. 31. Factor $x^2-2(a-n)x-4an$.

6. To factor a trinomial of the form ax^2+bx+c.

EXERCISES

Factor $3x^2+11x-4$.

SOLUTION. — If this trinomial is the product of two binomial factors, may be found by reversing the process of multiplication illustrated ercise 32, page 68.

nce $3x^2$ is the product of the *first terms* of the binomial factors, the terms, each containing x, are $3x$ and x.

nce -4 is the product of the last terms, § 84, they must have e signs, *and the only possible last terms* are 4 and -1, -4 and 1, or -2.

Hence, associating these pairs of factors of -4 with $3x$ and x in all possible ways, the possible binomial factors of $3x^2+11x-4$ are:

$$\left.\begin{matrix}3x+4\\x-1\end{matrix}\right\},\quad \left.\begin{matrix}3x-1\\x+4\end{matrix}\right\},\quad \left.\begin{matrix}3x-4\\x+1\end{matrix}\right\},\quad \left.\begin{matrix}3x+1\\x-4\end{matrix}\right\},\quad \left.\begin{matrix}3x+2\\x-2\end{matrix}\right\},\quad \left.\begin{matrix}3x-2\\x+2\end{matrix}\right\}.$$

Of these we select *by trial* the pair that will give $+11x$ (the middle term of the given trinomial) for the algebraic sum of the "cross-products," that is, the second pair.

$$\therefore 3x^2+11x-4=(3x-1)(x+4).$$

REMARK. — Since changing the signs of two factors of a number does not change the value of the number, $3x^2+11x-4$ has also the factors $(-3x+1)$ and $(-x-4)$; thus,

$$3x^2+11x-4=(-3x+1)(-x-4).$$

Such negative factors, however, are not usually required.

By a reversal of the law of signs for multiplication and from the above solution it may be observed that:

1. *When the sign of the last term of the trinomial is* $+$, *the last terms of the factors must be both* $+$ *or both* $-$, *and like the sign of the middle term of the trinomial.*

2. *When the sign of the last term of the trinomial is* $-$, *the sign of the last term of one factor must be* $+$, *and of the other* $-$.

Factor:

2. $5x^2+9x-2.$
3. $2x^2-5x-12.$
4. $3x^2-17x+10.$
5. $3x^2-7x-6.$
6. $6x^2-13x+6.$
7. $6x^2-11x-35.$

When the coefficient of x^2 is a square, and when the square root of the coefficient of x^2 is exactly contained in the coefficient of x, the trinomial may be factored as follows:

8. Factor $9x^2+30x+16.$

SOLUTION

$$\begin{aligned}&9x^2+30x+16\\&=(3x)^2+10(3x)+16\\&=(3x+2)(3x+8).\end{aligned}$$

9. Factor $4x^2-5x-6$.

SOLUTION

$$4x^2-5x-6=(4x^2-5x-6)\times\frac{4}{4}=\frac{16x^2-20x-24}{4}$$

$$=\frac{(4x)^2-5(4x)-24}{4}=\frac{(4x-8)(4x+3)}{4}$$

$$=\frac{4(x-2)(4x+3)}{4}=(x-2)(4x+3).$$

EXPLANATION. — Although the first term is a square, *its square root is contained exactly in the second term.* But if such a trinomial is mul- ied by the coefficient of x^2, the resulting trinomial will be one whose ond term exactly contains the square root of its first term.

Multiplying the given trinomial by 4, factoring as in exercise 8, and iding the result by 4, we find that the factors of the given trinomial $(x-2)$ and $(4x+3)$.

10. Factor $24x^2+14x-5$.

SUGGESTION. — *When the first term is not a square*, it may always be de a square whose square root will be contained exactly in the second m by *multiplying* the trinomial *by the coefficient of* x^2, but frequently a aller multiplier will accomplish the same result. In this case multiply 6, and divide by the same number to avoid changing the value of the pression.

Separate into simplest factors, testing results:

11. $2x^2+x-15$.
12. $9x^2-42x+40$.
13. $5x^2+13x+6$.
14. $25x^2+15x+2$.
15. $16x^2+20x-66$.
16. $36x^2-48x-20$.
17. $9x^2+43x-10$.
18. $25x^2+25x-24$.
19. $49x^2-42x-55$.
20. $16x^2+50x-21$.
21. $9x^4-10x^2-16$.
22. $27b^4-3b^2-14$.
23. $10x^6-2x^3-44$.
24. $2x^2+5xy+2y^2$.
25. $2x^2+3xy-2y^2$.
26. $3x^2-10xy+3y^2$.
27. $15x^2-14x-8$.
28. $15x^2+17x-4$.
29. $21a^2-a-10$.
30. $18x^2-3x-36$.

157. To factor the sum or the difference of two cubes.

By applying the principles of §§ 134–136,

$$\frac{a^3+b^3}{a+b}=a^2-ab+b^2 \text{ and } \frac{a^3-b^3}{a-b}=a^2+ab+b^2.$$

Then, § 121, $a^3+b^3=(a+b)(a^2-ab+b^2)$,

and $a^3-b^3=(a-b)(a^2+ab+b^2)$.

By use of these forms any expression that can be writt the sum or the difference of two cubes may be factored.

EXERCISES

158. 1. Factor x^6+y^6.

SOLUTION

$$x^6+y^6=(x^2)^3+(y^2)^3=(x^2+y^2)(x^4-x^2y^2+y^4).$$

2. Factor $a^9-125\,b^3$.

SOLUTION

$$a^9-125\,b^3=(a^3)^3-(5\,b)^3=(a^3-5\,b)(a^6+5\,a^3b+25\,b^2).$$

Factor, and test each result:

3. x^3+y^3.	9. $x-x^4$.	15. $r^{3x}-729\,s^{3x}$.
4. x^3-y^3.	10. $v^7+27\,v$.	16. $512\,x^{6n}+64\,y^{3n}$
5. m^3-1.	11. $a^3b^3-c^3d^3$.	17. $1+(a+b)^3$.
6. $1+m^3$.	12. $r^6+64\,s^3$.	18. $(x-y)^3-8$.
7. x^3-y^6.	13. $x^3y^6z^9-216$.	19. $8(m+n)^3+1$
8. r^9+s^9.	14. $343\,n^3+1000$.	20. $(x-y)^3-(x+$

159. To factor the sum or the difference of the same odd p of two numbers.

By applying §§ 134–136, as in § 157, any expression can be written as the sum or the difference of the sam powers of two numbers may be resolved into two factors.

$$a^5+b^5=(a+b)(a^4-a^3b+a^2b^2-ab^3+b^4),$$

and $$a^5-b^5=(a-b)(a^4+a^3b+a^2b^2+ab^3+b^4).$$

EXERCISES

60. 1. Factor $m^5 + 32x^5$.

SOLUTION. — $m^5 + 32x^5 = m^5 + (2x)^5$
$$= (m+2x)(m^4 - 2m^3x + 4m^2x^2 - 8mx^3 + 16x^4).$$

Factor $128a^{14} - 1$.

SOLUTION

$$28a^{14} - 1 = (2a^2)^7 - 1$$
$$= (2a^2 - 1)(64a^{12} + 32a^{10} + 16a^8 + 8a^6 + 4a^4 + 2a^2 + 1).$$

actor:

$m^5 + n^5$.	8. $1 + a^7$.	13. $m^7 - m^2n^5$.
$m^5 - n^5$.	9. $x^6 - y^3$.	14. $a^6b - ab^6$.
$x^7 - 1$.	10. $x^5 + y^{10}$.	15. $x^{5m} - y^{5n}$.
$x^9 + y^9$.	11. $a^5 + 32$.	16. $1 - a^5b^{10}c^{15}$.
$x^6 - x$.	12. $64 - 2a^5$.	17. $x^{10} + 243a^5$.

161. To factor the difference of the same even powers of two mbers.

EXERCISES

1. Factor $a^6 - b^6$.

FIRST SOLUTION

134–136, $a^6 - b^6 = (a-b)(a^5 + a^4b + a^3b^2 + a^2b^3 + ab^4 + b^5)$
$$= (a-b)(a^5 + a^2b^3 + a^4b + ab^4 + a^3b^2 + b^5)$$
$$= (a-b)[a^2(a^3+b^3) + ab(a^3+b^3) + b^2(a^3+b^3)]$$
$$= (a-b)(a^2+ab+b^2)(a^3+b^3)$$
57, $= (a-b)(a^2+ab+b^2)(a+b)(a^2-ab+b^2)$.

SECOND SOLUTION

2, $a^6 - b^6 = (a^3)^2 - (b^3)^2 = (a^3+b^3)(a^3-b^3)$
7, $= (a+b)(a^2-ab+b^2)(a-b)(a^2+ab+b^2)$.

Considering *the* even powers as squares, as in the second solution, *process may be* regarded as *factoring the difference of two squares.*

Separate into simplest factors:

2. $x^6 - y^6$.	5. $x^4 - 16$.	8. $1 - b^6$.
3. $x^6 - 1$.	6. $x^4 - 81$.	9. $64 - y^6$.
4. $a^8 - b^8$.	7. $a^4 - 625$.	10. $1 - x^6$.

162. All the preceding methods of finding binomial factors are really *special* methods. The following is a *general* method of finding binomial factors, when they exist.

163. To factor by the factor theorem.

Zero multiplied by any number is equal to 0.

Conversely, if a product is equal to zero, at least one of the factors must be 0 or a number equal to 0.

If $5x = 0$, since 5 is not equal to 0, x must equal 0.

If $5(x - 3) = 0$, since 5 is not equal to 0, x must have such a value as to make $x - 3$ equal to 0; that is, $x = 3$.

If $5(x - 3)$, or $5x - 15$, or any other polynomial in x reduces to 0 when $x = 3$, $x - 3$ is a factor of the polynomial.

Sometimes a polynomial in x reduces to 0 for more than one value of x. For example, $x^2 - 5x + 6$ equals 0 when $x = 3$ and also when $x = 2$; or when $x - 3 = 0$ and $x - 2 = 0$. In this case both $x - 3$ and $x - 2$ are factors of the polynomial.

$$\therefore x^2 - 5x + 6 = (x - 3)(x - 2).$$

164. Factor Theorem. — *If a polynomial in x, having positive integral exponents, reduces to zero when r is substituted for x, the polynomial is exactly divisible by $x - r$.*

The letter r represents any number that we may substitute for x.

PROOF. — Let D represent any rational integral expression containing x, and let D reduce to zero when r is substituted for x.

It is to be proved that D is exactly divisible by $x - r$.

Suppose that the dividend D is divided by $x - r$ until the remainder does not contain x. Denote the remainder by R and the quotient by Q.

Then, $$D = Q(x - r) + R. \quad (1)$$

But, since D reduces to zero when $x = r$, that is, when $x - r = 0$ (1) becomes $$0 = 0 + R; \text{ whence, } R = 0.$$

That is, the remainder is zero, and the division is exact.

EXERCISES

1. Factor $x^3 - x^2 - 4x + 4$.

SOLUTION

n $x = 1$, $\quad x^3 - x^2 - 4x + 4 = 1 - 1 - 4 + 4 = 0$.

efore, $x - 1$ is a factor of the given polynomial.

ling $x^3 - x^2 - 4x + 4$ by $x - 1$, the quotient is found to be $x^2 - 4$.

152, $\quad x^2 - 4 = (x + 2)(x - 2)$.

$$\therefore x^3 - x^2 - 4x + 4 = (x - 1)(x + 2)(x - 2).$$

ESTIONS. — 1. Only factors of the absolute term of the polynomial substituted for x in seeking factors of the polynomial of the form or if $x - r$ is one factor, the absolute term of the polynomial is duct of r and the absolute term of the other factor.

n substituting the factors of the absolute term, try them in order ng with the numerically smallest.

Then 1 is substituted for x, the value of the polynomial is equal to n of its coefficients; then $x-1$ is a factor when the sum of the nts is equal to 0.

Factor $17x^3 - 14x^2 - 37x - 6$.

SOLUTION

the sum of the coefficients is not equal to 0, $x - 1$ is not a factor.

n $x = -1$, $17x^3 - 14x^2 - 37x - 6 = -17 - 14 + 37 - 6 = 0$.

efore, $x - (-1)$, or $x + 1$, is a factor of the given polynomial.

ling $17x^3 - 14x^2 - 37x - 6$ by $x + 1$, the quotient is found to be $31x - 6$, which in turn may be tested for factors by the factor n.

tituting factors of -6 for x, it is found that:

n $x = 2$, $\quad 17x^2 - 31x - 6 = 68 - 62 - 6 = 0$.

efore, $x - 2$ is a factor of $17x^2 - 31x - 6$.

ling by $x - 2$, the other factor is found to be $17x + 3$.

$$\therefore 17x^3 - 14x^2 - 37x - 6 = (x + 1)(x - 2)(17x + 3).$$

Factor $2x^3 + x^2y - 5xy^2 + 2y^3$.

ESTION. — When $x = y$,

$$2x^3 + x^2y - 5xy^2 + 2y^3 = 2y^3 + y^3 - 5y^3 + 2y^3 = 0.$$

efore, $x - y$ is a factor of $2x^3 + x^2y - 5xy^2 + 2y^3$.

Factor by the factor theorem:

4. $x^2-31x+30.$
5. $4x^2-7x+3.$
6. $26x^2-10x-16.$
7. $48x^2-31x-17.$
8. $36x^2-61x+25.$
9. $x^3-9x^2+23x-15.$
10. $x^3-13x^2+47x-35.$
11. $x^3-14x^2+35x-22.$
12. $x^3-4x^2-7x+10.$
13. $x^3-6x^2-9x+14.$
14. $x^3-12x^2+41x-30.$
15. $x^3-11x^2+31x-21.$
16. $x^3-10x^2+29x-20.$
17. $x^3-16x^2+71x-56.$
18. $x^3-57x+56.$
19. $x^3-21xy^2+20y^3.$
20. $x^3-31xy^2-30y^3.$
21. $x^3-13xy^2+12y^3.$
22. $x^3-7x+6.$
23. $x^3-19x+30.$
24. $x^3-67x-126.$
25. $x^3-39x-70.$
26. $a^3+4a^2-11a-30.$
27. $a^3+9a^2+26a+24.$
28. $m^3-6m^2-m+30.$
29. $b^3-5b^2-29b+105.$
30. $a^3+10a^2-17a-66.$
31. $m^3+7m^2+2m-40.$
32. $b^3+16b^2+73b+90.$
33. $n^3+12n^2+41n+42.$
34. $x^4-15x^2+10x+24.$
35. $x^4-25x^2+60x-36.$
36. $x^4+13x^2-54x+40.$
37. $x^4+22x^2+27x-50.$
38. $x^4-9x^2y^2-4xy^3+12y^4.$
39. $x^4-9x^2y^2+12xy^3-4y^4.$
40. $x^4-x^3-7x^2+x+6.$
41. $x^4-9x^3+21x^2+x-30.$
42. $x^4+8x^3+14x^2-8x-15.$
43. $x^5-4x^4+19x^2-28x+12.$
44. $x^5-18x^3+30x^2-19x+30.$
45. $x^5-10x^4+40x^3-80x^2+80x-32.$

SPECIAL APPLICATIONS AND DEVICES

166. Factor:

1. $a^2+b^2+c^2+d^2+2ab-2ac+2ad-2bc+2bd-2cd.$

SOLUTION. — Since the polynomial consists of the squares of four numrs together with twice the product of each of them by each succeeding mber, the polynomial is the square of the sum of four numbers, § 111, d may be separated into two equal factors containing a, b, c, and d ith proper signs.

Since the ab, ad, and bd terms are *positive*, a and b, a and d, and b d d must have *like signs*; since the ac, bc, and cd terms are *negative*, and c, b and c, and c and d must have *unlike signs*.

Therefore, the factors are either

$$(a+b-c+d)(a+b-c+d)$$

$$(-a-b+c-d)(-a-b+c-d).$$

2. $9x^2+4y^2+25z^2-12xy+30xz-20yz.$
3. $25m^2+36n^2+p^2-60mn-10mp+12np.$
4. $a^2+16x^4+36y^2-8ax^2+12ay-48x^2y.$
5. $x^2+4a^2+b^2+y^2+4ax-2bx+2xy-4ab+4ay-2by.$
6. $m^2+4n^2+a^2+9-4mn-2am+6m+4an-12n-6a.$

167. The principle by which the difference of two squares factored has many special applications.

1. Factor $a^4+a^2b^2+b^4$.

SOLUTION. — Since $a^4+a^2b^2+b^4$ lacks $+a^2b^2$ of being a perfect square, l since the value of the polynomial will not be changed by adding a^2b^2 l also subtracting a^2b^2, the polynomial may be written

$$a^4+2a^2b^2+b^4-a^2b^2,$$

ich is the difference of two squares.

$$\therefore a^4+a^2b^2+b^4=a^4+2a^2b^2+b^4-a^2b^2$$
$$=(a^2+b^2)^2-a^2b^2$$
$$=(a^2+ab+b^2)(a^2-ab+b^2).$$

2. Factor $4a^4-13a^2+9$.

UGGESTION. $4a^4-13a^2+9=4a^4-12a^2+9-a^2=(2a^2-3)^2-a^2.$

3. Factor a^4+4.

SUGGESTION. $a^4+4=a^4+4a^2+4-4a^2=(a^2+2)^2-4a^2$.

Factor the following:

4. $x^4+x^2y^2+y^4$.
5. $a^8+a^4b^4+b^8$.
6. $9x^4+20x^2y^2+16y^4$.
7. $4a^4+11a^2b^2+9b^4$.
8. $16a^4-17a^2x^2+x^4$.
9. $25x^4-29x^2y^2+4y^4$.
10. x^4+x^2+1.
11. n^8+n^4+1.
12. $16x^4+4x^2y^2+y^4$.
13. $a^4b^4-21a^2b^2+36$.
14. $25a^4-14a^2b^4+b^8$.
15. $9a^4+26a^2b^2+25b^4$.
16. b^4+64.
17. a^4+4b^4.
18. m^8+4.
19. a^4+324.
20. a^8-16.
21. m^5+4mn^4.
22. x^4+64y^4.
23. $4a^4+81$.
24. $x^5y^2+4xy^2$.

168. Many polynomials may be written in the form x^2+px+q, x^2 and x being replaced by polynomials.

1. Factor $9x^2+4y^2+12z^2+21xz+14yz+12xy$.

SOLUTION.
$$\begin{aligned} & 9x^2+4y^2+12z^2+21xz+14yz+12xy \\ &= (9x^2+12xy+4y^2)+(21xz+14yz)+12z^2 \\ &= (3x+2y)^2+7z(3x+2y)+4z\cdot 3z \\ \text{§ 154,} \quad &= (3x+2y+4z)(3x+2y+3z). \end{aligned}$$

Factor the following:

2. $a^2+2ab+b^2+8ac+8bc+15c^2$.
3. $x^2-6xy+9y^2+6xz-18yz+5z^2$.
4. $m^2+n^2-2mn+7mp-7np-30p^2$.
5. $16n^2+55-64n-16m+m^2+8mn$.
6. $9m^4+k^2-30+39m^2+13k+6m^2k$.
7. $25a^2+y^2+10x^2+10ay-35ax-7xy$.
8. $a^2+b^2+c^2+2ab+2ac+2bc+5a+5b+5c+6$.

REVIEW OF FACTORING

ımmary of Cases. — In the previous pages the student ıd to factor expressions of the following types:

Monomials; as a^2b^3c. (§ 144)

Polynomials whose terms have a common factor; as

$$nx + ny + nz. \qquad (\S\ 146)$$

Polynomials whose terms may be grouped to show a com- ɔmial factor; as

$$ax + ay + bx + by. \qquad (\S\ 147)$$

Trinomials that are perfect squares; as

$$a^2 + 2\,ab + b^2 \text{ and } a^2 - 2\,ab + b^2. \qquad (\S\S\ 148\text{–}151)$$

Polynomials that are perfect squares; as

$$a^2 + b^2 + c^2 + 2\,ab + 2\,ac + 2\,bc. \qquad (\S\ 166)$$

The difference of two squares; as

$$a^2 - b^2 \qquad (\S\ 152)$$

$$a^4 + a^2b^2 + b^4. \qquad (\S\ 167)$$

Trinomials of the form

$$x^2 + px + q. \qquad (\S\ 154)$$

Trinomials of the form

$$ax^2 + bx + c \qquad (\S\ 156)$$

The sum or the difference of two cubes; as

$$a^3 + b^3 \text{ or } a^3 - b^3. \qquad (\S\ 157)$$

The sum or the difference of the same odd powers of two as

$$a^n + b^n \text{ or } a^n - b^n \text{ (when } n \text{ is odd)}. \qquad (\S\ 159)$$

The difference of the same even powers of two numbers; as

$$a^n - b^n \text{ (when } n \text{ is even)}. \qquad (\S\ 161)$$

***Polynomials* having binomial factors.** (§§ 162–165)

170. General Directions for Factoring Polynomials.—1. *move monomial factors if there are any.*

2. *Then endeavor to bring the polynomial under some on the cases II–XI.*

3. *When other methods fail, try the factor theorem.*

4. *Resolve into prime factors.*

Each factor should be divided out of the given expression as so found in order to simplify the discovery of the remaining factors.

171. Factor the following:

1. y^4-1.
2. $1-x^8$.
3. $x^{10}-1$.
4. x^6-1.
5. $a-a^7$.
6. b^7+b.
7. p^4+4.
8. $1+x^{12}$.
9. $y-a^4y$.
10. x^2y-y^3.
11. $a^{13}-ab^{12}$.
12. a^4-256.
13. $64-2y^5$.
14. $7n^7-7n$.
15. $4x^4-4x$.
16. $7y^4-175$.
17. $8-27a^3$
18. $32x-2$
19. $6b^4+24$.
20. a^5+27a
21. b^2-196.
22. $450-2a$
23. $4m^3+.0$
24. $125-8x$
25. $x^2-xy-132y^2$.
26. $ax^2-3ax-4a$.
27. x^3+5x^2-6x.
28. $3x^2+30x+27$.
29. $128a^2-250a^5$.
30. $5x^{10}+10x^5-15$.
31. $6x^2-19x+15$.
32. $x^{2n}+2x^ny^p+y^{2p}$.
33. $7x^2-77xy-84y^2$.
34. $y^2-25yx+136x^2$.
35. $9x^2-24xy+16y^2$.
36. $x^2-ax-72a^2$.
37. $n^2-an-90a^2$.
38. $a^2b^2+ab-56$.
39. $10a^2c+33ac-7$
40. $60ny^2-61ny-$
41. $25x^2+60xy+36$
42. $6ax^2+5axy-6$
43. $169x^4-26ax^3+$
44. $a^4c^4+a^2b^2c^2+b^4$.
45. $16x^4+4x^2y^2+y^4$.
46. b^4c-13b^2c+42

47. $17x^2+25x-18$.

48. $5x^2-26xy+5y^2$.

49. $y^2+16ay-36a^2$.

50. $8a^2-21ab-9b^2$.

51. $60a^2+8ax-3x^2$.

52. $30x^2-37x-77$.

53. $2x^3+28x^2+66x$.

54. $a^2+b^2-c^2-2ab$.

55. $ax^2+10ax-39a$.

56. $n^4+n^2a^2b^4+a^4b^8$.

57. $a^2z^4+a^2z^2+a^2$.

58. $a^2-16a-17$.

59. $a^2x^2-4ax+3$.

60. $b^8+b^4y^2+y^4$.

61. x^7-2x^6+x.

62. $x^3+x^2y-41xy^2-105y^3$.

63. $x^2-cx+2dx-2cd$.

64. $x^3y+4x^2y-31xy-70y$.

65. $x^2-3ax+4bx-12ab$.

66. $ax^3-9ax^2+26ax-24a$.

67. $12ax-8bx-9ay+6by$.

68. $25x^2-9y^2-24yz-16z^2$.

69. $x^2-z^2+y^2-a^2-2xy+2az$.

70. $2b^2m-3ab^2+2bmx-3abx$.

71. $a^2+b^2+c^2-2ab-2ac+2bc$.

72. $x^3y+14x^2y+43xy+30y$.

73. $x^3y-15x^2y+38xy-24y$.

74. $abx^3+3abx^2-abx-3ab$.

75. $3bmx+2bm-3anx-2an$.

76. $20ax^3-28ax^2+5a^2x-7a^2$.

77. $x^2+9y^2+25z^2-6xy-10xz+30yz$.

78. $9x^2+y^2+16z^2-6xy-8yz+24xz$.

79. $x^2y^2z^2+a^2b^2+1+2abxyz+2xyz+2ab$.

80. $a^2b^2+b^2c^2+c^2d^2-2ab^2c+2abcd-2bc^2d$.

81. $x^8+n^4x^4+n^8+2n^2x^6+2n^4x^4+2n^6x^2$.

82. $a^2b^2x^2-a^2b^2-b^2x^2+b^2-a^2x^2+a^2+x^2-1$.

83. $(a+b)^6-1$.

84. a^3-2a^2+1.

85. b^3-4b^2+8.

86. x^3-10x^2+125.

87. $8x^4-6x^2-35$.

88. $3x^6+96x$.

89. $(a-2)^3+(a-1)^3$.

90. $12x^3+3x^2-8x-2$.

91. $2x^2+10x+ax+5a$.

92. $x^3+5x^2-29x-105$.

93. $a^2b^2 - 4\,abx - 4\,x + 2\,ab + 4\,x^2$.

94. $(a+b)^2(x-y) - (a+b)(x^2-y^2)$.

95. $1 - x^2 + abx^2 + bx^3 - bx - ab$.

96. $x^2 - x^3 + x^2y - xy + x^2y - xy^2$.

97. $x^{2n-2} + b^2y^2 + 2\,x^{n-1}by$.

98. $x^3 + 15\,x^2 + 75\,x + 125$.

99. $4(ab+cd)^2 - (a^2+b^2-c^2-d^2)^2$.

100. $x^{3n} - a^{3b}$.

101. $(a^2+b^2-c^2)^2 - 4\,a^2b^2$.

102. $a^4b^2 + a^2b - 12$.

103. $x^3 - xy - x^2y + y^2$.

104. $x^4 - 4\,x^2y^2 + 2\,x^3 - 16\,y^3$.

105. $a^4 - b^4 - (a+b)(a-b)$.

106. $x^3 - 6\,x^2 + 12\,x - 8$.

107. $1000\,x^3 - 27\,y^3$.

108. $(a+x)^4 - x^4$.

109. $1 + (x+1)^3$.

110. $ab - bx^n + x^ny^m - ay^m$.

111. $x^9 + 4\,x$.

112. $x^5 - x^2 - x^4 + x^3$.

113. $(a+b)^4 - (b-c)^4$.

114. $3\,ab(a+b) + a^3 + b^3$.

115. $(x+y)^3 + (x-y)^3$.

116. $a^3 - (a+b)^3$.

117. $x^4 - 119\,x^2y^2 + y^4$.

118. $m^3 + m^2 - mn - mn^2$.

119. $(x^2-y^2)^2 - (x^2-xy)^2$.

120. $x^6 - y^6 - 3\,x^2y^2(x^2-y^2)$.

121. $(x^2+6x+9)^2 - (x^2+5x+6$

122. $2 - 3\,b + 3\,ab - 2\,a + 4\,a^2 - 6\,a^2b$.

123. Factor $32 - x^5$ by the factor theorem.

124. Factor $16 + 5\,x - 11\,x^2$ by the factor theorem.

125. If n is odd, factor $x^n - a^n$ by the factor theorem.

126. If n is odd, factor $x^n + r^n$ by the factor theorem.

127. Factor $x^3 - 6\,bx^2 + 12\,b^2x - 8\,b^3$ by the factor theoren

128. Discover by the factor theorem for what values of *between 1 and 20,* $x^n + a^n$ has no binomial factors.

EQUATIONS SOLVED BY FACTORING

.72. 1. Find the values of x in $x^2+1=10$.

FIRST PROCESS

$+1=10$

$x^2 = 9$

$x \cdot x = 3 \cdot 3 \qquad \therefore x = 3$

$x \cdot x = -3 \cdot -3 \therefore x = -3$

$\therefore x = \pm 3$

EXPLANATION. — On transposing the known term 1 to the second member, the first member contains the second power, only, of the unknown number. On separating each member into two equal factors,

$x \cdot x = 3 \cdot 3$ or $x \cdot x = -3 \cdot -3$.

Since, if $x = 3$, $x \cdot x = 3 \cdot 3$, and if -3, $x \cdot x = -3 \cdot -3$, the value of x that makes $x^2 = 9$, or that makes $\cdot 1 = 10$, is either $+3$ or -3; that is, $x = \pm 3$.

'ind the two values of x in each of the following:

2. $x^2+3=28$.
3. $x^2+1=50$.
4. $x^2-5=59$.
5. $x^2-7=29$.
6. $x^2+ 3= 84$.
7. $x^2-24=120$.
8. $x^2+11=180$.
9. $x^2-11=110$.

). Find the values of x in $x^2+1=10$.

SECOND PROCESS

$$x^2+1=10$$
$$x^2-9=0$$
$$(x-3)(x+3)=0$$
$-3=0$, whence $x=3$

$+3=0$, whence $x=-3$

$$\therefore x=\pm 3$$

EXPLANATION. — The first process is given in exercise 1.

In the second process, all terms are brought to the first member, which is factored as the difference of the squares of two numbers.

Since the product of the two factors is 0, one of them is equal to 0. Therefore, $x-3=0$ or $x+3=0$; ıce, $x=3$ or $x=-3$; that is, $x=\pm 3$.

ılve for x, and verify results:

11. $x^2+35=39$.
12. $x^2-50=50$.
13. $x^2+90=91$.
14. $x^2-31^2=0$.
15. $x^2-4b^2=0$.
16. $x^2-9n^2=0$.

17. $x^2 - 21 = 4.$

18. $x^2 - 56 = 8.$

19. $x^2 - 3a^2 = 6a^2.$

20. $x^2 + 5b^4 = 6b^4.$

21. $x^2 - 40 = 24.$

22. $32 - x^2 = 28.$

23. $65 - x^2 = 16.$

24. $4x^2 - 8b^2 = 8$

25. $x^2 + 25 = 25 +$

26. $x^2 - 30 = 2(2b$

27. Solve $x^2 + 2am = a^2 + m^2.$

SOLUTION

$$x^2 + 2am = a^2 + m^2.$$

$$x^2 = a^2 - 2am + m^2.$$

$$x \cdot x = (a - m)(a - m)$$

or

$$x \cdot x = -(a - m) \cdot -(a - m).$$

$$\therefore x = \pm (a - m).$$

Solve for x, and verify:

28. $x^2 - c^2 = d^2 - 2cd.$

29. $x^2 - b^2 = 4bc + 4c^2.$

30. $x^2 - n^2 = 6n + 9.$

31. $x^2 + 10a = a^2 + 25.$

32. $x^2 - a^2 = 2a + 1.$

33. $x^2 - m^2 = 8m + 16.$

34. $x^2 - c^2 = 36 - 1$

35. $x^2 - 4b^2 = 36 -$

36. $x^2 - a^2 = 9 - 6$

37. $x^2 - b^4 = 4 - 4$

38. $x^2 - a^2b^2 = 2ab$

39. $x^2 - r^4 = b^4 - 2$

40. Find the values of x in $x^2 + 4x = 45.$

FIRST PROCESS

$$x^2 + 4x = 45$$

$$x^2 + 4x - 45 = 0$$

$$(x - 5)(x + 9) = 0$$

$$\therefore x - 5 = 0 \text{ or } x + 9 = 0$$

$$\therefore x = 5 \text{ or } -9$$

SECOND PROCE

$$x^2 + 4x = 45$$

$$x^2 + 4x + 4 = 49$$

$$(x + 2)(x + 2) = 7 \cdot 7$$

$$\therefore x + 2 = 7 \text{ or}$$

$$\therefore x = 5 \text{ o}$$

EXPLANATION. — For the first process the explanation is similar to that given for exercise 10.

In the second process, it is seen that, by adding 4 to each member of the equation, the first member will become the square of the binomial $(x+2)$. On solving for $(x+2)$ as for x in previous exercises, $x+2=\pm 7$; whence, $x=\pm 7-2=+7-2$ or $-7-2=5$ or -9.

SUGGESTION. — In the following exercises, when the coefficient of the first power of the unknown number is *even*, either of the above processes may be used; but when it is *odd*, the first process is simpler.

Solve, and verify results:

41. $x^2-6x=40$.
42. $x^2-8x=48$.
43. $x^2-5x=-4$.
44. $x^2+4x+3=0$.
45. $r^2+6r+8=0$.
46. $x^2-9x+20=0$.
47. $x^2-3x=40$.
48. $x^2-9x=36$.
49. $x^2+11x=26$.
50. $x^2-12x=45$.
51. $y^2-15y=54$.
52. $y^2-21y=46$.
53. $x^2-10x=96$.
54. $y^2-20y=96$.
55. $y^2+12y=85$.
56. $y^2+42=13y$.
57. $t^2+63=16t$.
58. $v^2-60=11v$.
59. $x^2-7x=18$.
60. $x^2+10x=56$.
61. $x^2+12x=28$.
62. $n^2+11n+30=0$.
63. $x^2+x-132=0$.
64. $32=4w+w^2$.
65. $3s=88-s^2$.
66. $160=x^2-6x$.
67. $4y=y^2-192$.
68. $600=y^2-10y$.
69. $c^2+16c-36=0$.
70. $l^2+15l-34=0$.

Solve for x, y, or z, and verify results:

71. $x^2+2bx+b^2=0$.
72. $z^2+4az+4a^2=0$.
73. $x^2-(a+b)x+ab=0$.
74. $x^2+(c+d)x+cd=0$.

75. $x^2+(a+2)x+2a=0.$

76. $y^2-(c-n)y-nc=0.$

77. $x^2-(a-d)x-ad=0.$

78. $x^2-(b+7)x+7b=0.$

79. $(2x+3)(2x-5)-(3x-1)(x-2)=1.$

80. $(2x-6)(3x-2)-(5x-9)(x-2)=4.$

81. Solve $6x^2+5x-21=0.$

Solution

$$6x^2+5x-21=0.$$

Factoring, § 156, $$(2x-3)(3x+7)=0.$$

$$\therefore 2x-3=0$$

or $$3x+7=0.$$

$$\therefore x=\tfrac{3}{2} \text{ or } -\tfrac{7}{3}.$$

Solve, and verify results:

82. $3x^2+2x-1=0.$

83. $5x^2+4x-1=0.$

84. $3y^2+y-10=0.$

85. $3y^2-4y-4=0.$

86. $4y^2+9y-9=0.$

87. $7x^2+6x-1=0.$

88. $2v^2-9v-35=0.$

89. $6y^2-22y+20=0.$

90. $3x^2+13x-30=0.$

91. $4x^2+13x-12=0.$

92. Solve the equation $x^3-2x^2-5x+6=0.$

Solution

$$x^3-2x^2-5x+6=0.$$

Factoring, § 163, $$(x-1)(x-3)(x+2)=0.$$

$$\therefore x-1=0 \text{ or } x-3=0 \text{ or } x+2=0;$$

whence, $$x=1 \text{ or } 3 \text{ or } -2.$$

93. $x^3-15x^2+71x-105=0.$

94. $x^3+10x^2+11x-70=0.$

95. $x^3-12x+16=0.$

96. $x^3-19x-30=0.$

97. $x^4+x^3-21x^2-x+20=0.$

98. $x^4-7x^3+x^2+63x-90=0.$

99. $x^5-11x^4+45x^3-85x^2+74x-24=0.$

HIGHEST COMMON FACTOR

173. The sum of the exponents of the literal factors of a ...tional integral term determines the **degree of the term.**

Thus, a and $5\,a$ are of the first degree ; $3\,x^2$ and $3\,xy$ are of the second ...gree ; $4\,ab^3c$ and $x^3(y-1)^2$ are of the fifth degree.

174. The term of highest degree in any rational integral ...pression determines the **degree of the expression.**

Thus, the expression $x^3 - 6\,x^2 + 11\,x - 6$ is of the third degree.

175. An expression that is a factor of each of two or more ...pressions is called **a common factor** of them.

176. The common factor of two or more expressions that ...s the largest numerical coefficient and is of the highest ...gree is called their **highest common factor** (H. C. F.).

The common factors of $4\,a^3b^2$ and $6\,a^2b$ are 2, a, b, a^2, $2\,a$, $2\,b$, $2\,a^2$, ab, ...b, a^2b, and $2\,a^2b$, with sign $+$ or $-$. Of these, $2\,a^2b$ (or $-\,2\,a^2b$) has ... largest numerical coefficient and is of the highest degree, and is there...e the highest common factor.

The highest common factor may be either positive or negative, but ...ally only the positive sign is taken.

The *highest* common factor, or divisor, of $4\,a^3b^2$ and $6\,a^2b$ is $2\,a^2b$, ...ardless of the values that a and b may represent. What the arith...tical *greatest* common divisor is depends upon the values of a and b. ...$a = 2$ and $b = 6$,

...H. C. F. $= 2\,a^2b = 48$; but since $4\,a^3b^2 = 1152$ and $6\,a^2b = 144$, the ...thmetical *greatest* common divisor $= 144$.

177. Principle. — *The highest common factor of two or more ...pressions is equal to the product of all their common prime ...tors.*

178. Expressions that have no common prime factor, except ...are *said to be* **prime to each other.**

EXERCISES

179. 1. Find the H. C. F. of $12\, a^4b^2c$ and $32\, a^2b^3c^3$.

SOLUTION

The arithmetical greatest common divisor or highest common fact of 12 and 32 is 4. The highest common factor of a^4b^2c and $a^2b^3c^3$ is a^2b^2 Hence, H. C. F. $= 4\, a^2b^2c$.

RULE. — *To the greatest common divisor of the numeric coefficients annex each common literal factor with the least exp nent it has in any of the expressions.*

Find the highest common factor of:

2. $10\, x^3y^2$, $10\, x^2y^3$, and $15\, xy^4z$.
3. $70\, a^6b^3$, $21\, a^4b^4$, and $35\, a^4b^5$.
4. $8\, m^7n^3$, $28\, m^6n^4$, and $56\, m^5n^2$.
5. $4\, b^3cd$, $6\, b^2c^2$, and $24\, abc^3$.
6. $3(a+b)^2$ and $6(a+b)^3$.
7. $6(a+b)^2$ and $4(a+b)(a-b)$.
8. $12(a-x)^3$, $6(a-x)^2$, and $(a-x)^4$.
9. $30(x+y)^2$, $18(x+y)$, and $(x+y)^3$.
10. $10(x-y)^4z^3$ and $15(z-y)(x-y)^3$.
11. $3(a^2-b^2)^2$ and $a(a-b)(a^2-b^2)$.
12. What is the H.C.F. of $3\, x^3 - 3\, xy^2$ and $6\, x^3 - 12\, x^2y + 6\, xy$

PROCESS

$$\begin{aligned} 3\, x^3 - 3\, xy^2 \quad &= 3\, x(x+y)(x-y) \\ 6\, x^3 - 12\, x^2y + 6\, xy^2 &= 2 \cdot 3\, x\,(x-y)(x-y) \\ \hline \therefore \text{H. C. F.} &= 3\, x(x-y) \end{aligned}$$

EXPLANATION. — For convenience in selecting the common factors, t expressions are resolved into their simplest factors.

Since the only common prime factors are 3, x, and $(x-y)$, the high *common factor* sought (§ 177) is their product, $3\, x\,(x-y)$.

Find the highest common factor of:

13. $x^2 - 2x - 15$ and $x^2 - x - 20$.

14. $x^4 - y^4$, $x^2 - y^2$, and $x + y$.

15. $a^2 + 7a + 12$ and $a^2 + 5a + 6$.

16. $x^3 + y^3$ and $x^2 + 2xy + y^2$.

17. $a^2 - x^2$ and $a^2 - 2ax + x^2$.

18. $a^2 - b^2$ and $a^2 + 2ab + b^2$.

19. $x^4 + x^2y^2 + y^4$ and $x^2 + xy + y^2$.

20. $x^3 + y^3$, $x^5 + y^5$, and $x^2y + xy^2$.

21. $a^4 + a^2b^4 + b^8$ and $3a^2 - 3ab^2 + 3b^4$.

22. $a^2 - x^2$, $a^2 + 2ax + x^2$, and $a^3 + x^3$.

23. $ax - y + xy - a$ and $ax^2 + x^2y - a - y$.

24. $a^2b - b - a^2c + c$ and $ab - ac - b + c$.

25. $1 - 4x^2$, $1 + 2x$, and $4a - 16ax^2$.

26. $(a - b)(b - c)$ and $(c - a)(a^2 - b^2)$.

27. $24x^3y^3 + 8x^2y^3$ and $8x^3y^3 - 8x^2y^3$.

28. $6x^2 + x - 2$ and $2x^2 - 11x + 5$.

29. $16x^2 - 25$ and $20x^2 - 9x - 20$.

30. $x^4 + xy^3$ and $x^2y + xy^2$.

31. $pq^4 + p^6q$ and $qp^3 + q^2p^2$.

32. $17abc^3d^5 - 51a^3bc^4d^4$ and $abc^2d^2 - 3a^3bc^3d$.

33. $38xyz - 95x^3yz^2$ and $34xy^2z - 85x^2yz^2$.

34. $x^7y + xy^4$ and $2x^5y - 2x^3y^2 + 2xy^3$.

35. $6r^7 + 10r^6s - 4r^5s^2$ and $2r^7 + 2r^6s - 4r^5s^2$.

36. $x^4 - x^3 - 2x^2$, $x^4 - 2x^3 - 3x^2$, and $x^4 - 3x^3 - 4x^2$.

37. $3m^3n^3 - 3mn^4$ and $6m^4n^3 + 6m^3n^2 - 6m^2n^3 - 6mn^4$.

38. $7l^3t^3 + 35l^2t^3 + 42lt^3$ and $7l^4t^3 + 21l^3t^3 - 28l^2t^3 - 84lt^3$.

39. $x^2 + a^2 - b^2 + 2ax$, $x^2 - a^2 + b^2 + 2bx$, and $x^2 - a^2 - b^2 - 2ab$.

Apply the factor theorem when necessary.

40. x^2-6x+5 and x^3-5x^2+7x-3.

41. x^2-4 and $x^3-10x^2+31x-30$.

42. x^3-4x+3 and $x^3+x^2-37x+35$.

43. $3x^4-12x^2$ and $6x^4+30x^3-96x^2+24x$.

44. $a^4b-a^2b^3$ and $a^4b+2a^3b^2+2a^2b^3+ab^4$.

45. $9-n^2$ and n^2-n-6.

SUGGESTION. — Change $9-n^2$ to $-(n^2-9)=-(n+3)(n-3)$.

46. $1-x^2$ and $x^3-6x^2-9x+14$.

47. $4-a^2$ and $a^4+a^3-10a^2-4a+24$.

48. $(9-x^2)^2$ and $x^4+5x^3-3x^2-45x-54$.

SUGGESTION. —$(9-x^2)^2=(x^2-9)^2$.

49. $(4-c^2)^2$ and $c^3+9c^2+26c+24$.

50. $(x-x^2)^3$, $(x^2-1)^3$, and $(1-x)^3$.

51. $(1-y^4)^2$ and $(y+1)^2(1-y)^2(y^3-7y+6)$.

52. $xy-y^2$, $-(y^3-x^2y)$, and x^2y-xy^2.

53. $16-s^4$, $2s-s^2$, and s^2-4s+4.

54. y^4-x^4, x^5+y^5, and $y^2+2yx+x^2$.

55. $x^2-(y+z)^2$, $(y-x)^2-z^2$, and $y^2-(x-z)^2$.

56. $(y-x)^2(n-m)^3$ and $(x^2y-y^3)(m^2n-2mn^2+n^3)$.

When some of the given expressions are difficult to fact their factors may often be discovered by dividing by those the more easily factored expressions.

57. $(m+2)(m^2-9)$ and $m^4-3m^3+3m^3n+3m^2n^2-9m^2n$ $mn^3-9mn^2-3n^3$.

58. $6x^2-3x-45$, $9x^2-33x+18$, and $6x^3-3x^2-39x-$

59. $2x^4-x^3-x^2$, $2x^2+x-3$, and x^3-x^2-x+1.

60. $x^4-4x^3+2x^2+x+6$, $2x^3-9x^2+7x+6$, and x^2-5x+

61. s^3-8, s^3+s^2+2s-4, and $s^4+2s^3-s^2-10s-20$.

62. x^5+y^5 and $x^5-2x^4y+2x^3y^2-2x^2y^3+2xy^4-y^5$.

LOWEST COMMON MULTIPLE

180. An expression that exactly contains each of two or lore given expressions is called a **common multiple** of them.

$6\,abx$ is a common multiple of a, $3\,b$, $2\,x$, and $6\,abx$. These numbers ay have other common multiples, as $12\,abx$, $6\,a^2b^2x$, $18\,a^3bx^2$, etc.

181. The expression having the smallest numerical coefficient nd of *lowest degree* that will exactly contain each of two or ore given expressions is called their **lowest common multiple.**

$6\,abx$ is the lowest common multiple (L. C. M.) of a, $3\,b$, $2\,x$, and $6\,abx$. ie *lowest* common multiple of a and b is ab, regardless of the values that and b may represent. What the arithmetical *least* common multiple is pends upon the values of a and b. If $a = 6$ and $b = 2$, the least common ultiple is not 12, the value of ab, but 6.

The lowest common multiple may have either sign.

In §§ 180, 181, only *rational integral* expressions are included.

182. PRINCIPLE. — *The lowest common multiple of two or more xpressions is the product of all their different prime factors, each actor being used the greatest number of times it occurs in any of ie expressions.*

EXERCISES

183. 1. What is the L. C. M. of $12x^2yz^4$, $6\,a^2xy^2$, and $8\,axyz^2$?

SOLUTION. — The lowest common multiple of the numerical coefficients found as in arithmetic. It is 24.

The literal factors of the lowest common multiple are each letter with ie highest exponent it has in any of the given expressions (Prin.). They re, therefore, a^2, x^2, y^2, and z^4.

The product of the numerical and literal factors, $24\,a^2x^2y^2z^4$, is the *west common multiple* of the given expressions.

2. What is the L. C. M. of $x^2-2xy+y^2$, y^2-x^2, and x^3+y^3?

PROCESS

$$\begin{array}{ll} x^2-2xy+y^2 & =(x-y)(x-y) \\ y^2-x^2=-(x^2-y^2) & =-(x+y)(x-y) \\ x^3+y^3 & =(x+y)(x^2-xy+y^2) \\ \hline \text{L. C. M.} & =(x-y)^2(x+y)(x^2-xy+y^2) \\ & =(x-y)^2(x^3+y^3) \end{array}$$

RULE. — *Factor the expressions into their prime factors.*

Find the product of all their different prime factors, using each factor the greatest number of times it occurs in any of the given expressions.

The factors of the L. C. M. may often be selected without separating the expressions into their *prime* factors.

Find the lowest common multiple of:

3. a^3x^2y, a^2xy^3, and ax^2y.
4. $10a^2b^2c^2$, $5ab^2c$, and $25b^3c^3d^3$.
5. $16a^2b^3c$, $24c^3de$, and $36a^4b^2d^2e^3$.
6. $18a^2br^2$, $12p^2q^2r$, and $54ab^2p^3q$.
7. x^my^2, $x^{m-1}y^3$, $x^{m-3}y^4$, and $x^{m+1}y$.
8. x^2-y^2 and $x^2+2xy+y^2$.
9. x^2-y^2 and $x^2-2xy+y^2$.
10. x^2-y^2, $x^2+2xy+y^2$, and $x^2-2xy+y^2$.
11. a^2-n^2 and $3a^3+6a^2n+3an^2$.
12. x^4-1 and $a^2x^2+a^2-b^2x^2-b^2$.
13. a^2+1, $ab-b$, a^2+a, and $1-a^2$.
14. $2x+y$, $2xy-y^2$, and $4x^2-y^2$.
15. $1+x$, $x-x^2$, $1+x^2$, and $x^2(1-x)$.

16. $2x+2$, $5x-5$, $3x-3$, and x^2-1.

17. $16b^2-1$, $12b^2+3b$, $20b-5$, and $2b$.

18. $1-2x^2+x^4$, $(1-x)^2$, and $1+2x+x^2$.

19. $xy-y^2$, x^2+xy, $xy+y^2$, and x^2+y^2.

20. y^3-x^3, x^2+xy+y^2, and x^2-xy.

21. b^2-5b+6, $b^2-7b+10$, and $b^2-10b+16$.

22. x^2+7x-8, x^2-1, $x+x^2$, and $3ax^2-6ax+3a$.

23. x^2-a^2, $a-2x$, a^2+2ax, and $a^3-3a^2x+2ax^2$.

24. m^3-x^3, m^2+mx, m^2+mx+x^2, and $(m+x)x^2$.

25. $2-3x+x^2$, x^2+4x+4, x^2+3x+2, and $1-x^2$.

26. x^2-y^2, $x^4+x^2y^2+y^4$, x^3+y^3, and x^2+xy+y^2.

27. $x^3+x^2y+xy^2+y^3$ and $x^3-x^2y+xy^2-y^3$.

28. a^2+4a+4, a^2-4, $4-a^2$, and a^4-16.

29. $a^2-(b+c)^2$, $b^2-(c+a)^2$, and $c^2-(a+b)^2$.

30. $m-n$, $(m^2-n^2)^2$, and $(m+n)^3$.

31. a^6-b^3 and $a^8+a^4b^2+b^4$.

32. x^6+y^6 and $a^2x^2-b^2y^2+a^2y^2-b^2x^2$.

33. a^4-a^2+1, a^6+1, a^4+a^2+1, and a^2-1.

34. $2(ax^2-x^3)^2$, $3x(a^2x-x^3)^3$, and $6(a^2x^2-a^4)$.

35. $(yz^2-xyz)^2$, $y^2(xz^2-x^3)$, and $x^2z^2+2xz^3+z^4$.

SUGGESTION. — In solving the following, use the factor theorem.

36. $x^3-6x^2+11x-6$ and $x^3-9x^2+26x-24$.

37. $x^3-5x^2-4x+20$ and $x^3+2x^2-25x-50$.

38. x^3-4x^2+5x-2 and $x^3-8x^2+21x-18$.

39. x^3+5x^2+7x+3 and $x^3-7x^2-5x+75$.

40. x^3+2x^2-4x-8, $x^3-x^2-8x+12$, $x^3+4x^2-3x-18$.

FRACTIONS

184. A fraction is *expressed* by two numbers, one called numerator, written above a line, and the other the **denomin** written below the line.

If a and b represent positive integers, as 3 and 4, the frac $\frac{a}{b}$ is equal to $\frac{3}{4}$; that is, it represents 3 of the 4 equal part anything. This is the arithmetical notion of a fraction.

But, since a and b may be any numbers, positive or nega integral or fractional, $\frac{a}{b}$ may represent an expression like Since a thing cannot be divided into $5\frac{3}{8}$ equal parts, algel fractions are not described accurately by the definition monly given in arithmetic. But, since an expression lik regarded as 20 fourths, is equivalent to 5, or $20 \div 4$, it is dent that the numerator of a fraction may be regarded dividend, and the denominator as its divisor; and this i pretation of a fraction is broad enough to include the fra $\frac{a}{b}$ when a and b represent any numbers whatever. Hence,

The expression of an unexecuted division, in which the divi is the numerator and the divisor the denominator, is an **alge fraction.**

The fraction $\frac{a}{b}$ is read, '*a divided by b.*'

185. The numerator and denominator of a fraction are c its **terms.**

186. An expression, some of whose terms are integral some fractional, is called a **mixed number,** or a **mixed expre**

$a - \frac{a-b}{c}$, $\frac{x^2}{a^2} - 2 + \frac{a^2}{x^2}$, and $a - b + \frac{1}{ab}$ are mixed expressions

Signs in Fractions

187. The sign written before the dividing line of a fraction is called the **sign of the fraction.**

It belongs to the fraction as a *whole,* and not to the numerator or to the denominator alone.

In $-\frac{x}{3z}$ the sign of the fraction is $-$, while the signs of x and $3z$ are $+$.

188. An expression like $\frac{-a}{-b}$ indicates a process in division, in which the quotient is to be found by dividing a by b and prefixing the sign according to the law of signs in division; that is,

$$\frac{-a}{-b}=+\frac{a}{b}, \qquad \frac{+a}{+b}=+\frac{a}{b},$$

$$\frac{-a}{+b}=-\frac{a}{b}, \qquad \frac{+a}{-b}=-\frac{a}{b}.$$

By observing the above fractions and their values the following principles may be deduced:

189. Principles.—1. *The signs of both the numerator and the denominator of a fraction may be changed without changing the sign of the fraction.*

2. *The sign of either the numerator or the denominator of a fraction may be changed, provided the sign of the fraction is changed.*

When either the numerator or the denominator is a polynomial, its sign changed by changing the sign of each of its terms. Thus, the sign of $-b$ is changed by writing it $-a+b$, or $b-a$.

EXERCISES

190. Reduce to fractions having positive numbers in both terms:

1. $\frac{-3}{-4}$.
2. $\frac{2}{-5}$.
3. $\frac{-a-x}{2x}$.
4. $\frac{-4c}{-b-y}$.
5. $-\frac{-a-b}{c+d}$.
6. $-\frac{-2}{-a-y}$.
7. $-\frac{-2-m}{2+n}$.
8. $\frac{-4(a+b)}{5(-x-y)}$.

191. By the law of signs for multiplication, the product of *two* negative factors is *positive;* of *three* negative factors, *negative;* of *four* negative factors, *positive;* and so on. Hence,

PRINCIPLES. — 3. *The sign of either term of a fraction is changed by changing the signs of an odd number of its factors.*

4. *The sign of either term of a fraction is not changed by changing the signs of an even number of its factors.*

EXERCISES

192. 1. Show that $\dfrac{(a-b)(d-c)}{(c-a)(b-c)}=\dfrac{(a-b)(c-d)}{(a-c)(b-c)}$.

SOLUTION OR PROOF

Changing $(d-c)$ to $(c-d)$ changes the sign of *one* factor of the numerator and therefore changes the sign of the numerator (Prin. 3).

Similarly, changing $(c-a)$ to $(a-c)$ changes the sign of the denominator (Prin. 3).

We have changed the signs of both terms of the fraction. Therefore, the sign of the fraction is not affected (Prin. 1).

2. Show that $\dfrac{(b-a)(d-c)}{(c-b)(a-c)}=-\dfrac{(a-b)(c-d)}{(b-c)(a-c)}$.

SOLUTION OR PROOF

Changing the signs of *two* factors of the numerator does not change the sign of the numerator (Prin. 4).

Changing the sign of *one* factor of the denominator changes the sign of the denominator (Prin. 3).

Since we have changed the sign of only *one* term of the fraction, we must change the sign of the fraction (Prin. 2).

3. Show that $\dfrac{-b}{b-a}$ may be properly changed to $\dfrac{b}{a-b}$.

4. From $\dfrac{-a}{b-a+c}$ derive $\dfrac{a}{a-b-c}$ by proper steps.

5. Prove that $\dfrac{3}{1-x}=-\dfrac{3}{x-1}$; that $-\dfrac{2}{4-x^2}=\dfrac{2}{x^2-4}$.

6. Prove that $\dfrac{2}{a(b-a)} = -\dfrac{2}{a(a-b)}$.

7. Prove that $\dfrac{5x}{(x+y)(y-x)} = -\dfrac{5x}{(x+y)(x-y)}$.

8. Prove that $-\dfrac{2a}{9-a^2} = \dfrac{2a}{(a+3)(a-3)}$.

9. Prove that $\dfrac{1}{(b-a)(c-b)} = \dfrac{1}{(a-b)(b-c)}$.

10. Prove that $\dfrac{(m-n)(m+n)}{(a-c)(b-a)} = \dfrac{-m^2+n^2}{(a-c)(a-b)}$.

11. Prove that $\dfrac{(a-b)(b-a+c)}{(y-x)(z-y)(z-x)} = \dfrac{(a-b)(a-b-c)}{(x-y)(y-z)(x-z)}$.

REDUCTION OF FRACTIONS

193. The student will find no difficulty with algebraic fractions, if he will bear in mind that they are essentially the same as the fractions he has met in arithmetic. He will have occasion to change fractions to higher or lower terms; to write integral and mixed expressions in fractional form; to change fractions to integers or mixed numbers; to add, subtract, multiply, and divide with algebraic fractions just as he has learned to do with arithmetical fractions.

194. The process of changing the form of an expression without changing its value is called **reduction**.

195. PRINCIPLE. — *Multiplying or dividing both terms of a fraction by the same number does not change the value of the fraction;* that is,

$$\frac{a}{b} = \frac{am}{bm} \text{ or } \frac{am}{bm} = \frac{a}{b}.$$

196. *A fraction is in its* **lowest terms** *when its terms are prime to each other.*

197. To reduce fractions to higher or lower terms.

EXERCISES

1. Reduce $\frac{a}{a+b}$ to a fraction whose denominator is a^2-b^2.

PROCESS

$$(a^2-b^2)\div(a+b)=a-b.$$

Then, $$\frac{a}{a+b}=\frac{a(a-b)}{(a+b)(a-b)}=\frac{a^2-ab}{a^2-b^2}.$$

EXPLANATION. — Since the required denominator is $(a-b)$ times the given denominator, in order that the value of the fraction shall not be changed (§ 195) both terms of the fraction must be multiplied by $(a-b)$.

2. Reduce $\frac{5a}{6}$ to a fraction whose denominator is 42.

3. Reduce $\frac{3x}{11b}$ to a fraction whose denominator is $55b$.

4. Reduce $\frac{3a}{14x}$ to a fraction whose denominator is $84xy$.

5. Reduce $\frac{4a^2}{5y}$ to a fraction whose denominator is $20y^3$.

6. Reduce $\frac{x-3}{x-1}$ to a fraction whose denominator is $(x-1)^2$.

7. Reduce $\frac{2x-5}{2x+5}$ to a fraction whose denominator is $(2x+5)^2$.

8. Reduce $\frac{a}{3-a}$ to a fraction whose numerator is $3a+a^2$.

9. Reduce $\frac{x-y}{2x+y}$ to a fraction whose numerator is x^2-y^2.

10. Reduce $\frac{-1}{x-2}$ to a fraction whose denominator is $4-x^2$.

11. Reduce $x-5$ to a fraction whose denominator is $x+5$.

12. Reduce $\dfrac{21\,a^2x^2y}{30\,a^3xz}$ to its lowest terms.

PROCESS

$$\frac{a^2x^2y}{a^3xz}=\frac{7\,xy}{10\,az}$$

EXPLANATION. — Since a fraction is in its lowest terms when its terms are prime to each other, the given fraction may be reduced to its lowest terms by removing in succession all common factors of its numerator and denominator (§ 195), as 3, a, a, and x; or by dividing the terms by their highest common factor, $3\,a^2x$.

Reduce to lowest terms:

13. $\dfrac{16\,m^2nx^2z^2}{40\,am^3yz^3}$.

14. $\dfrac{210\,bc^2d}{750\,ab^2c}$.

15. $\dfrac{35\,a^2bcd^3}{42\,ab^2cd^4}$.

16. $\dfrac{77\,a^7x^5b^3y}{121\,a^3b^5c^7}$.

17. $\dfrac{-25\,x^2y^5z^2}{-100\,x^4y^3}$.

18. $\dfrac{a^2xy^2}{a^3xy}$.

19. $\dfrac{m^3n^3}{am^2n^4}$.

20. $\dfrac{a^2b^2x^2}{b^3xy^2}$.

21. $\dfrac{x^{m+2}y^2}{x^my^4}$.

22. $\dfrac{x^m}{x^{m-2}a^2}$.

23. $\dfrac{x^{m+1}y}{xy^{m+1}}$.

24. $\dfrac{x^{m-n+1}}{ax}$.

25. $\dfrac{a^xb^y}{3\,a^{2x}b}$.

26. $\dfrac{a^{m+2r}y^{2r}}{2\,a^ry^{4r}}$.

27. $\dfrac{ax^{m-n+1}}{bx^{m-n}}$.

28. $\dfrac{a^2-b^2}{a^2+2\,ab+b^2}$.

29. $\dfrac{a^2-2\,ab+b^2}{a^2-b^2}$.

30. $\dfrac{4\,a^2-9\,x^2}{8\,a^3+27\,x^3}$.

31. $\dfrac{3\,a^2+3\,ab}{a^4+ab^3}$.

32. $\dfrac{3\,x^2y-6\,xy}{x^4y-8\,xy}$.

33. $\dfrac{3\,a^2b-3\,b^3}{2\,a^3b-2\,b^4}$.

34. $\dfrac{2\,x^2y^2-8\,y^4}{4\,x^3y-32\,y^4}$.

35. $\dfrac{10\,nx+10\,ny}{25\,nx^2-25\,ny^2}$.

36. $\dfrac{x^{n+2}-x^n}{x^{n+3}-x^n}$.

37. $\dfrac{a^{n+4}-a^ny^4}{a^{n+3}+a^{n+1}y^2}$.

38. $\dfrac{x^4y+x^2y^3+y^5}{x^6-y^6}$.

39. $\dfrac{x^4y-x^2y^3+y^5}{x^6+y^6}$.

40. $\dfrac{a^2-11a+24}{a^2-a-6}$.

41. $\dfrac{x^3-6x^2+5x}{x^3+2x^2-35x}$.

42. $\dfrac{7x-2x^2-3}{2x^2+7x-4}$.

43. $\dfrac{a(a+2b)^4}{b(a^2-4b^2)^2}$.

44. $\dfrac{a^3+2a^2b+ab^2}{a^5-2a^3b^2+ab^4}$.

45. $\dfrac{x^2-2x^4+x^6}{x^6-x^2}$.

46. $\dfrac{x^3+5x^2-6x}{2x^2-2}$.

47. $\dfrac{x^3-7x+6}{x^4-10x^2+9}$.

48. $\dfrac{20-21x+x^3}{x^4-26x^2+25}$.

49. $\dfrac{x^3+3x^2+3x+1}{4+4x-x^2-x^3}$.

50. $\dfrac{a^3-3a^2b+3ab^2-b^3}{3ab^2-3a^2b}$.

51. $\dfrac{3a^2+4ax-4x^2}{9a^2-12ax+4x^2}$.

52. $\dfrac{2ax-ay-4bx+2by}{4ax-2ay-2bx+by}$.

53. $\dfrac{9x^3-13a^2x-4a^3}{3bx+3xy-4ab-4ay}$.

54. $\dfrac{m-m^2-n+mn}{m-mn+n^2-n}$.

55. $\dfrac{am-an-m+n}{am-an+m-n}$.

56. $\dfrac{x^3+5x^2-9x-45}{x^3+3x^2-25x-75}$.

57. $\dfrac{x^3+2x^2-23x-60}{x^3-11x^2-10x+200}$.

58. $\dfrac{3x^3-7x^2+4}{5x^3-17x^2+16x-4}$.

59. $\dfrac{x^3-6x^2y+2xy^2+3y^3}{5y^3+2xy^2-6x^2y-x^3}$.

60. $\dfrac{a^2+b^2+2c^2+2ab+3ac+3bc}{a^2+b^2+c^2+2ab+2ac+2bc}$.

61. $\dfrac{a^2+b^2+c^2+2ab-2ac-2bc}{a^2+b^2-c^2+2ab}$.

62. $\dfrac{a^2+b^2+c^2-2ab-2ac+2bc}{a^2+b^2+5c^2-2ab-6ac+6bc}$.

63. $\dfrac{4a^2+9b^2+16c^2+12ab+16ac+24bc}{4a^2-9b+16c^2+16ac}$.

198. To reduce a fraction to an integral or a mixed expression.

EXERCISES

1. Reduce $\frac{ax+b}{x}$ to a mixed number.

PROCESS

$$\frac{ax+b}{x}=a+\frac{b}{x}$$

EXPLANATION. — Since a fraction may be regarded as an expression of unexecuted division, by performing the division indicated the fraction is changed into the form of a mixed number.

2. Reduce $\frac{a^3-3a^2-a+1}{a^2}$ to a mixed number.

SOLUTION. $\frac{a^3-3a^2-a+1}{a^2}=a-3+\frac{-a+1}{a^2}=a-3-\frac{a-1}{a^2}.$

The division should be continued until the remainder is lower in degree than the denominator or no longer contains the denominator.

Reduce to an integral or a mixed expression:

3. $\frac{4x^3-8x^2+2x-1}{2x}.$

4. $\frac{ab-bc-cd+d^2}{b}.$

5. $\frac{a^2x^3-ax^2-x-1}{ax}.$

6. $\frac{x^2-x-15}{x-4}.$

7. $\frac{x^3-2xy-y^3}{x-y}.$

8. $\frac{x^2-6xy+4y^2}{2xy}.$

9. $\frac{x^3-6x^2+14x-9}{x-2}.$

10. $\frac{x^4-3x^2+5x-1}{x-3}.$

11. $\frac{x^3+5x^2+3x-6}{x+2}.$

12. $\frac{a^3+9a^2+24a+22}{a+3}.$

13. $\frac{x^4+2x^3-x^2+5}{x^3+x^2}.$

14. $\frac{4a^5+12a^3-a^2+34}{2a^3+5}.$

15. $\frac{a^2+3ab-5b^2+bc}{a-2b}.$

16. $\frac{x^3-7x-4x+40}{x^2-3}.$

17. $\frac{a^3+3a^2b-ab^2+ab}{a^2+b}.$

18. $\frac{x^2-xy-3y^2-z}{x+y}.$

19. $\dfrac{a^4+3a^2b^2+b^4}{a^2+b^2}$.

20. $\dfrac{4x^2+22x+21}{2x+4}$.

21. $\dfrac{x^4+4x^3y+6x^2y^2+4xy^3}{x+y}$.

22. $\dfrac{m^5-2m^3n-3mn^2-2mn}{m^3-mn}$.

199. To reduce dissimilar fractions to similar fractions.

200. Fractions that have the same denominator are called **similar fractions.**

201. Fractions that have different denominators are called **dissimilar fractions.**

202. Principle. — *The lowest common denominator of two or more fractions is the lowest common multiple of their denominators.*

The abbreviation L. C. D. is used instead of *lowest common denominator*.

EXERCISES

203. 1. Reduce $\dfrac{a}{3bc}$ and $\dfrac{c}{6ab}$ to fractions having their lowest common denominator.

PROCESS

$$\frac{a}{3bc}=\frac{a\times 2a}{3bc\times 2a}=\frac{2a^2}{6abc}$$

$$\frac{c}{6ab}=\frac{c\times c}{6ab\times c}=\frac{c^2}{6abc}$$

Explanation. — Since the L. C. D. of the given fractions is the lowest common multiple of their denominators (Prin.), the lowest common multiple of their denominators must be found. This is $6abc$.

To reduce the fractions to equivalent fractions having the common denominator $6abc$, the terms of each fraction (§ 195) must be multiplied by the quotient of $6abc$ divided by the denominator of that fraction.

Rule. — *Find the lowest common multiple of the denominators of the fractions for the lowest common denominator.*

Divide this denominator by the denominator of the first fraction, and multiply the terms of the fraction by the quotient.

Proceed in a similar manner with each of the other fractions.

All fractions should first be reduced to lowest terms.

2. Reduce $2m$ and $\frac{m+n}{m-n}$ to fractions having their L. C. D.

SUGGESTION. — First write $2m$ as a fraction with the denominator 1.

Reduce to similar fractions having their L. C. D.:

3. $\frac{x}{2}$ and $\frac{3y}{5}$.

4. $\frac{2a}{5b}$ and $3x$.

5. $\frac{a^2b}{c^2d}$ and $\frac{ab^2}{cd^2}$.

6. $\frac{3}{x^2y^4}, \frac{-6}{x^3y^3}, \frac{3}{x^4y^2}$.

7. $\frac{m-n}{a}, 2, \frac{a}{m+n}$.

8. $\frac{x^2}{x^2-1}, \frac{x}{x+1}, \frac{x}{x-1}$.

9. $\frac{a^3}{a^4-16}, \frac{a}{a^2+4}, \frac{2a}{4-a^2}$.

10. $\frac{4a}{a-b}, \frac{3b}{b+a}, \frac{1}{a^2-b^2}$.

11. $\frac{1}{x^2+7x+10}, \frac{1}{x^2+x-2}, \frac{1}{x^2+4x-5}$.

12. $\frac{a+5}{a^2-4a+3}, \frac{a-2}{a^2-8a+15}, \frac{a+1}{a^2-6a+5}$.

ADDITION AND SUBTRACTION OF FRACTIONS

204. It has been learned in arithmetic that *only similar fractions may be united into one fraction by addition or subtraction.*

The method of adding and subtracting similar fractions is much the same in algebra as in arithmetic. In algebra, however, subtraction of fractions practically reduces to addition of fractions, for every fraction to be subtracted is really added with its sign changed (§ 56, Prin.).

The usual method of changing the sign of a fraction, in such cases, is to *change the sign of its numerator* (§ 189, Prin. 2).

Thus, $$\frac{a}{x}+\frac{b}{x}-\frac{c}{x}+\cdots=\frac{a+b-c+\cdots}{x}.$$ That is,

205. PRINCIPLE. — *If fractions have a common denominator, their sum is the sum of their numerators divided by the common denominator.*

EXERCISES

206. 1. Add $\frac{3x}{4}$, $\frac{7x}{10}$, and $\frac{5y}{12}$.

PROCESS

$$\frac{3x}{4}+\frac{7x}{10}+\frac{5y}{12}=\frac{45x}{60}+\frac{42x}{60}+\frac{25y}{60}$$
$$=\frac{87x+25y}{60}.$$

EXPLANATION. — Since the fractions are dissimilar, they must be made similar before they can be united into one term. The L. C. D. $=60$.

Then, $\frac{3x}{4}=\frac{45x}{60}$; $\frac{7x}{10}=\frac{42x}{60}$; and $\frac{5y}{12}=\frac{25y}{60}$.

The sum $=\frac{45x}{60}+\frac{42x}{60}+\frac{25y}{60}=\frac{45x+42x+25y}{60}=\frac{87x+25y}{60}$.

2. Subtract $\frac{x-2}{7}$ from $\frac{5x-1}{8}+\frac{x}{4}$.

SOLUTION

$$\frac{5x-1}{8}+\frac{x}{4}-\frac{x-2}{7}=\frac{35x-7}{56}+\frac{14x}{56}-\frac{8x-16}{56}$$
$$=\frac{35x-7+14x-(8x-16)}{56}$$
$$=\frac{35x-7+14x-8x+16}{56}$$
$$=\frac{41x+9}{56}.$$

SUGGESTION. — When a fraction is preceded by the sign $-$, it is well for the beginner to inclose the numerator in a parenthesis, if it is a polynomial, as shown above.

RULE. — *Reduce the fractions to similar fractions having their lowest common denominator.*

Change the signs of all the terms of the numerators of fractions preceded by the sign $-$, then find the sum of the numerators, and write it over the common denominator.

Reduce the resulting fraction to its lowest terms, if necessary.

Add:

3. $\dfrac{2x}{5}$ and $\dfrac{3x}{2}$.

4. $\dfrac{4a}{3}$ and $\dfrac{6b}{5}$.

5. $\dfrac{2a}{3b}$ and $\dfrac{3a}{2b}$.

6. $\dfrac{-5}{7x}$ and $\dfrac{-2}{3x}$.

Subtract:

7. $\dfrac{5m}{6}$ from $\dfrac{4m}{3}$.

8. $\dfrac{4x}{9}$ from $-\dfrac{x}{2}$.

9. $-\dfrac{y}{3}$ from $\dfrac{3}{y}$.

10. $\dfrac{a+b}{3}$ from $\dfrac{a-b}{2}$.

Simplify:

11. $\dfrac{2x+1}{3}+\dfrac{x-2}{4}-\dfrac{x-3}{6}+\dfrac{5-x}{2}$.

12. $\dfrac{x-2}{6}-\dfrac{x-4}{9}+\dfrac{2-3x}{4}-\dfrac{2x+1}{12}$.

13. $\dfrac{x-1}{3}-\dfrac{x-2}{18}-\dfrac{4x-3}{27}+\dfrac{1-x}{6}$.

14. $\dfrac{2-6x}{5}+\dfrac{4x-1}{2}-\dfrac{5x-3}{6}-\dfrac{1-x}{3}$.

15. $\dfrac{x+3}{4}-\dfrac{x-2}{5}+\dfrac{x-4}{10}-\dfrac{x+3}{6}$.

16. $\dfrac{1-2a}{5}+\dfrac{2a-1}{4}-\dfrac{2a-a^2+1}{8}$.

17. $\dfrac{3+x-x^2}{4}-\dfrac{1-x+x^2}{6}-\dfrac{1-2x-2x^2}{3}$.

18. Reduce $\dfrac{5a^2+b^2}{a^2-b^2}-2$ to a fraction.

Solution

$$\frac{5a^2+b^2}{a^2-b^2}-\frac{2}{1}=\frac{5a^2+b^2-2(a^2-b^2)}{a^2-b^2}$$

$$=\frac{5a^2+b^2-2a^2+2b^2}{a^2-b^2}$$

$$=\frac{3(a^2+b^2)}{a^2-b^2}.$$

Reduce the following mixed expressions to fractions:

19. $a+\dfrac{b}{2}$.

20. $x-\dfrac{y}{2}$.

21. $\dfrac{a^2-c^2}{c}+5c$.

22. $\dfrac{1-x}{3}-4x$.

23. $a-\dfrac{a^2-ab}{b}$.

24. $a-\dfrac{a-b-c}{2}$.

25. $a+x-\dfrac{x^2}{a-x}$.

26. $a^2-ab+b^2-\dfrac{b^3}{a+b}$

Perform the additions and subtractions indicated:

27. $\dfrac{a-b}{ab}+\dfrac{b-c}{bc}$.

28. $\dfrac{a+b}{a-b}+\dfrac{a-b}{a+b}$.

29. $\dfrac{b-c}{bc}-\dfrac{a-c}{ac}$.

30. $\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}$.

31. $x+y-\dfrac{x^2+y^2}{x-y}$.

32. $\dfrac{x}{x-2}-\dfrac{x-2}{x+2}$.

33. $\dfrac{3a}{5}+\dfrac{b}{2}-3+\dfrac{1}{b}$.

34. $x+1+\dfrac{x^3-3}{x-1}$.

35. $m-\dfrac{m^2+n^2}{m-n}+n$.

36. $1-\dfrac{ax-bx+ab}{x^2}$.

37. $\dfrac{1}{x}+1+\dfrac{2x}{1+x}-2$.

38. $2a-3b-\dfrac{4a^2+9}{2a+3}$

39. $3a-2x-\dfrac{8a^2-4}{3a+2}$

40. $\dfrac{1}{x-1}-\dfrac{1}{x+1}-\dfrac{2}{x^2}$.

41. $\dfrac{1}{a+b}-\dfrac{1}{a-b}+\dfrac{2a}{a^2-b^2}$.

42. $\dfrac{a+x}{a-x}+\dfrac{a-x}{a+x}+\dfrac{4ax}{a^2-x^2}$.

43. $\dfrac{a+1}{a^2+a+1}+\dfrac{a-1}{a^2-a+1}$.

44. $3x+\dfrac{5}{ax}-\left(2x+\dfrac{3}{ax}\right)$.

45. $\dfrac{a-b}{2(a+b)}+\dfrac{a^2+b^2}{a^2-b^2}-\dfrac{a}{a-b}$.

46. $\dfrac{a+33}{a^2-9}-\dfrac{6}{a-3}+\dfrac{10}{a+3}$.

47. $\dfrac{a}{a-2}-\dfrac{a-2}{a+2}+\dfrac{3}{4-a^2}.$

ESTION.—By Prin. 1, § 189, $\dfrac{3}{4-a^2}=\dfrac{-3}{a^2-4}.$

48. $\dfrac{a+1}{a-1}+\dfrac{2}{a+1}+\dfrac{4a}{1-a^2}.$

49. $\dfrac{5x+2}{x^2-4}+\dfrac{2}{x-2}-\dfrac{3}{2-x}.$

50. $\dfrac{x(a+x)}{a-x}-\dfrac{3ax-x^2}{x-a}+4a.$

51. $\dfrac{1}{a^3+8}-\dfrac{1}{8-a^3}+\dfrac{1}{4-a^2}.$

52. $\dfrac{5(x-3)}{x^2-x-2}-\dfrac{2(x+2)}{x^2+4x+3}-\dfrac{x-1}{6-x-x^2}.$

53. Simplify $\dfrac{x^2+x+1}{x^2-x+1}-1+\dfrac{2x}{x^2+x+1}.$

SOLUTION

ducing the first fraction to a mixed number,

$$\frac{c+1}{c+1}-1+\frac{2x}{x^2+x+1}=1+\frac{2x}{x^2-x+1}-1+\frac{2x}{x^2+x+1}$$
$$=\frac{2x}{x^2-x+1}+\frac{2x}{x^2+x+1}=\frac{4x(x^2+1)}{x^4+x^2+1}.$$

ESTION.—Frequently, by reducing one or more of the given frac- mixed numbers, the integers cancel each other and the numerators lified.

54. $\dfrac{a^2+2ab+b^2}{a^2+b^2}-1+\dfrac{2ab}{a^2-b^2}.$

55. $\dfrac{a^2+3ab+2b^2}{a^2+3ab-4b^2}-\dfrac{a^2-13b^2}{a^2-16b^2}.$

56. $\dfrac{x+1}{x-1}+\dfrac{x-1}{x+1}-\dfrac{x+2}{x-2}-\dfrac{x-2}{x+2}.$

57. $\dfrac{x+3}{x-3}-\dfrac{x-3}{x+3}+\dfrac{x+4}{x-4}-\dfrac{x-4}{x+4}.$

58. $\dfrac{a}{a-b}-\dfrac{a}{a+b}-\dfrac{2\,ab}{a^2+b^2}-\dfrac{4\,ab^3}{a^4+b^4}.$

SUGGESTION. — Combine the first two fractions, then the result an third fraction, then this result and the fourth fraction.

59. $\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}-\dfrac{4\,ab}{a^2+b^2}+\dfrac{8\,ab^3}{a^4+b^4}.$

60. $\dfrac{1}{a-b}-\dfrac{1}{a+b}-\dfrac{2\,b}{a^2+b^2}+\dfrac{2\,b^3}{a^4+b^4}.$

61. $\dfrac{a+x}{a-x}+\dfrac{a^2+x^2}{a^2-x^2}-\dfrac{a-x}{a+x}-\dfrac{a^2-x^2}{a^2+x^2}-\dfrac{4\,a^3x+4\,ax^3}{a^4-x^4}.$

62. $\dfrac{x+y}{(y-z)(z-x)}-\dfrac{y+z}{(x-z)(x-y)}+\dfrac{z+x}{(y-x)(z-y)}.$

SOLUTION

$$\text{Sum} = \frac{x+y}{(y-z)(z-x)}+\frac{y+z}{(z-x)(x-y)}+\frac{z+x}{(x-y)(y-z)}$$

$$= \frac{(x^2-y^2)+(y^2-z^2)+(z^2-x^2)}{(x-y)(y-z)(z-x)}$$

$$= \frac{0}{(x-y)(y-z)(z-x)} = 0.$$

63. $\dfrac{1}{(b-c)(a-c)}+\dfrac{1}{(c-a)(a-b)}+\dfrac{1}{(b-a)(b-c)}.$

64. $\dfrac{a+1}{(a-b)(a-c)}+\dfrac{b+1}{(b-c)(b-a)}+\dfrac{c+1}{(a-c)(b-c)}.$

65. $\dfrac{c^2ab}{(c-a)(b-c)}-\dfrac{b^2ca}{(b-a)(b-c)}-\dfrac{a^2bc}{(a-b)(a-c)}.$

66. $\dfrac{b-c}{(b-a)(a-c)}-\dfrac{c-a}{(b-c)(a-b)}-\dfrac{a+b}{(a-c)(b-c)}.$

67. $\dfrac{c+a}{(a-b)(b-c)}-\dfrac{b+c}{(c-a)(b-a)}+\dfrac{a+b}{(c-b)(a-c)}.$

MULTIPLICATION OF FRACTIONS

207. Fractions are multiplied in algebra just as they are in arithmetic.

Thus, $$\frac{3}{4}\times\frac{5}{2}=\frac{3\times 5}{4\times 2}.$$

In general, $$\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}.$$ That is,

PRINCIPLE. — *The product of two or more fractions is equal to the product of their numerators divided by the product of their denominators.*

EXERCISES

208. 1. Multiply $\frac{x-5}{x+5}$ by x^2-25.

SOLUTION

$$\frac{x-5}{\cancel{x+5}}\cdot\frac{\overset{x-5}{\cancel{x^2-25}}}{1}=(x-5)^2=x^2-10x+25.$$

2. Multiply $\frac{x+3}{x+2}$ by $1+\frac{1}{x+1}$.

SOLUTION

$$\left(\frac{x+3}{x+2}\right)\left(1+\frac{1}{x+1}\right)=\frac{x+3}{x+2}\left(\frac{x+1}{x+1}+\frac{1}{x+1}\right)$$
$$=\frac{x+3}{\cancel{x+2}}\cdot\frac{\cancel{x+2}}{x+1}$$
$$=\frac{x+3}{x+1}.$$

GENERAL SUGGESTIONS. — 1. Any integer may be written with the denominator 1.

2. After finding the product of the numerators and the product of the denominators the resulting fraction may be reduced to lowest terms, in many cases, by canceling common factors from numerator and denominator. It is, however, more convenient to remove the common factors before performing the multiplications.

3. *Generally, mixed numbers should be reduced to fractions.*

Multiply:

3. $\dfrac{3ab}{4xy}$ by $\dfrac{2y}{3a^2}$.

4. $\dfrac{5xy}{2ac}$ by $\dfrac{3ax}{10y^3}$.

5. $\dfrac{4ab}{10c^2}$ by $\dfrac{3bc}{a^3}$.

6. $\dfrac{4mn}{3xy}$ by $-\dfrac{15bx}{16m^2}$.

7. $\dfrac{2ax}{12by}$ by $-\dfrac{10b^2}{x^2}$.

8. $\dfrac{a^m b^n}{4x}$ by $\dfrac{6x^2}{a^{m-1}b^{2n}}$.

9. $\dfrac{a^{m+1}}{b^{m+2}}$ by $\dfrac{b^{m+1}}{a^m}$.

10. $\dfrac{a}{a+b}$ by $\dfrac{b}{a-b}$.

11. $\dfrac{xy^2}{20-8x}$ by $\dfrac{25-10x}{x^2y}$.

12. $\dfrac{1-6x+5x^2}{x^2-3x+2}$ by $\dfrac{2-x}{1-x}$.

Simplify each of the following:

13. $\dfrac{(a-b)^2}{a+b} \times \dfrac{b}{a^2-ab} \times \dfrac{(a+b)^2}{a^2-b^2}$.

14. $\dfrac{a^4-x^4}{a^3+x^3} \times \dfrac{a+x}{a^2-x^2} \times \dfrac{a^2-ax+x^2}{(a+x)^2}$.

15. $\dfrac{4a-b}{2x+y} \times \dfrac{2a}{4a^2-ab} \times \dfrac{4x^2-y^2}{4}$.

16. $\dfrac{p+2}{x-3} \times \dfrac{3x^2-27}{2p^2-8} \times \dfrac{4}{px+3p}$.

17. $\dfrac{p^4-q^4}{(p-q)^2} \times \dfrac{p-q}{p^2+pq} \times \dfrac{p^2}{p^2+q^2}$.

18. $\dfrac{a^3+8}{a^3-8} \times \dfrac{a^2+2a+4}{a^2-2a+4}$.

19. $\dfrac{a^4+a^2x^2+x^4}{a^4-ax^3} \times \dfrac{x}{a^2-ax+x^2}$.

20. $\dfrac{a^4+4}{a^4+a^2+1} \times \dfrac{a^2+a+1}{a^2+2a+2}$.

$$\left(1-\frac{x-1}{x^2+6x+5}\right)\left(1-\frac{2}{x^2+7x+12}\right).$$

$$\left(1+\frac{7x+11}{x^2-4x-21}\right)\left(1-\frac{17x-11}{x^2+7x+10}\right).$$

$$\frac{a^2+ab+ac+bc}{ax-ay-x^2+xy}\cdot\frac{a^2-ax+ay-xy}{a^2+ac+ax+cx}\cdot\frac{x^2-x(y-a)-ay}{a^2-a(y-b)-by}.$$

$$\frac{x^3-5x^2+8x-4}{x^3-8x^2+19x-12}\cdot\frac{x^3-10x^2+33x-36}{x^3-6x^2+11x-6}.$$

$$\frac{x^4-3x^3-23x^2+75x-50}{x^4-5x^3-21x^2+125x-100}\cdot\frac{x^3-10x^2+29x-20}{x^3-12x^2+45x-50}.$$

DIVISION OF FRACTIONS

. The **reciprocal** of a number is 1 divided by the number.

reciprocal of 5 is $\frac{1}{5}$; of b, $\frac{1}{b}$; of $(a+b)$, $\frac{1}{a+b}$.

. Since $\frac{1}{d}$ is contained d times in 1, $\left(c \text{ times } \frac{1}{d}\right)$ is con- l $\frac{1}{c}$ of d times, or $\frac{d}{c}$ times, in 1; that is,

$$1\div\frac{c}{d}=\frac{d}{c}.$$

INCIPLE. — *The reciprocal of a fraction is the fraction ed.*

.. Since $1\div\frac{c}{d}=\frac{d}{c}$, and $a=1\cdot a$,

lows that $a\div\frac{c}{d}=\frac{d}{c}\cdot a$ or $a\cdot\frac{d}{c}$.

INCIPLE. — *Dividing by a fraction is equivalent to multi- by its reciprocal.*

EXERCISES

212. Divide:

1. 1 by $\frac{ab}{x}$.
2. 1 by $\frac{x^3}{x^2}$.
3. 1 by $\frac{a-b}{a+b}$.
4. 1 by $\frac{a-3}{a+3}$.

Write the reciprocal of:

5. $\frac{a}{b}$.
6. $\frac{3m}{p}$.
7. $\frac{m}{n}$.
8. $\frac{1}{3m}$.
9. $\frac{a-}{b-}$
10. $\frac{4}{ab}$.

11. Divide $\frac{x^2-4}{x^2-1}$ by $\frac{x+2}{x-1}$.

SOLUTION

$$\frac{x^2-4}{x^2-1} \div \frac{x+2}{x-1} = \frac{\cancel{(x+2)}(x-2)}{(x+1)\cancel{(x-1)}} \cdot \frac{\cancel{x-1}}{\cancel{x+2}} = \frac{x-2}{x+1}.$$

Simplify:

12. $\frac{5mn}{6bx} \div \frac{10m^2n}{3ax^2}$.
13. $\frac{12a^4b}{25ac} \div \frac{4ax}{24c^2}$.
14. $\frac{3abm}{7} \div abx$.
15. $\frac{my-y^2}{(m+y)^2} \div \frac{y^2}{m^2-y^2}$.
16. $\frac{(a-b)^2}{a+b} \div \frac{a^2-ab}{b}$.
17. $(4a+2) \div \frac{2a+1}{5a}$.
18. $\frac{a^4-b^4}{a^2-2ab+b^2} \div$
19. $\frac{x^3+y^3}{x^2-y^2} \div \frac{x^2+x}{x-}$
20. $\frac{m^4x+m^5}{m^3x-mx^3} \div \frac{m^3}{m^3}$
21. $\left(x \div \frac{1}{y}\right) \div \left(y^2 \div\right.$
22. $\left(\frac{a^3}{b} \div b^2\right) \div \left(\frac{a^2}{b^2} \times\right.$
23. $(a+c) \div \left(\frac{a^2-c^2}{1+x}\right.$

24. $\dfrac{a^2+b^2-c^2+2ab}{a^2-b^2-c^2+2bc} \div \dfrac{a^2-b^2+c^2-2ac}{a^2-b^2+c^2+2ac}$.

25. $\dfrac{x^3-6x^2+11x-6}{x^3+2x^2-19x-20} \div \dfrac{x^3-13x+12}{x^3+10x^2+29x+20}$.

26. $\left(y-x+\dfrac{x^2}{y}\right) \div \left(\dfrac{x}{y^2}+\dfrac{y}{x^2}\right)$.

SUGGESTION. — Reduce the dividend to a fraction.

27. $\left(x^2-\dfrac{1}{x^2}\right) \div \left(x-\dfrac{1}{x}\right)$.

28. $\left(1-\dfrac{y^2}{x^2}\right) \div \left(1-\dfrac{2x}{y}+\dfrac{x^2}{y^2}\right)$.

29. $\left(1+\dfrac{1}{y^2}+\dfrac{1}{y^4}\right) \div \left(1+\dfrac{1}{y}+\dfrac{1}{y^2}\right)$.

30. $\left(x-4+\dfrac{9}{x+2}\right) \div \left(1-\dfrac{4x-7}{x^2-4}\right)$.

31. $\left(x+\dfrac{3x+6}{x^2-1}+2\right) \div \left(x+3+\dfrac{1}{x+1}\right)$.

Complex Fractions

213. A fraction one or both of whose terms contains a fraction is called a **complex fraction.**

It is simply an expression of unexecuted division.

EXERCISES

214. 1. Simplify the expression $\dfrac{\frac{a}{b}}{\frac{x}{y}}$.

SOLUTION.
$$\frac{\frac{a}{b}}{\frac{x}{y}} = \frac{a}{b} \div \frac{x}{y} = \frac{a}{b} \times \frac{y}{x} = \frac{ay}{bx}.$$

Simplify:

2. $\dfrac{\dfrac{x+y}{ab}}{\dfrac{x^2-y^2}{ab^2}}$.

3. $\dfrac{a+\dfrac{b}{c}}{b+\dfrac{c}{a}}$.

4. $\dfrac{m-\dfrac{3m}{x}}{x-\dfrac{x}{m}}$.

5. $\dfrac{2+\dfrac{3a}{4b}}{a+\dfrac{8b}{3}}$.

6. $\dfrac{1-\dfrac{y^2}{x^2}}{1+\dfrac{y^2}{x^2}}$.

7. $\dfrac{x-\dfrac{1}{x}}{1+\dfrac{1}{x}}$.

8. $\dfrac{b-\dfrac{c}{2}}{\dfrac{b}{2}-c}$.

9. $\dfrac{ax-\dfrac{x^2}{2}}{\dfrac{a^2}{2}-ax}$.

10. $\dfrac{\dfrac{x+y}{y}-\dfrac{x+}{x}}{\dfrac{1}{y}-\dfrac{1}{x}}$

11. Simplify the expression $\dfrac{\dfrac{x^2}{y^2}-\dfrac{x}{y}+1}{\dfrac{x^2}{y^2}+\dfrac{x}{y}+1}$.

Solution. — On multiplying the numerator and denominator of fraction by y^2, which is the L. C. D. of the fractional parts of the merator and denominator, the expression becomes $\dfrac{x^2-xy+y^2}{x^2+xy+y^2}$.

Simplify:

12. $\dfrac{\dfrac{x^2-1}{x}}{\dfrac{x+1}{x^2}}$.

13. $\dfrac{\dfrac{1}{x}+\dfrac{1}{y+z}}{\dfrac{1}{x}-\dfrac{1}{y+z}}$.

14. $\dfrac{\dfrac{x^3+y^3}{xy}}{\dfrac{x^2-xy+y^2}{xy}}$.

15. $\dfrac{\dfrac{1}{a+1}}{1-\dfrac{1}{a+1}}$.

16. $\dfrac{\dfrac{x^2+y^2}{2y}-x}{\dfrac{x}{y}-\dfrac{y}{x}}$

17. $-\dfrac{\dfrac{1}{1-a}}{\dfrac{a}{a-1}}$.

Simplify:

18. $\dfrac{x-2+\dfrac{1}{x+2}}{x+2+\dfrac{1}{x-2}}$.

20. $\dfrac{6a-1-\dfrac{1}{a}}{\dfrac{2a-1}{3a}}$.

19. $\dfrac{\dfrac{1}{x}+\dfrac{4}{x^2}+\dfrac{4}{x^3}}{1+\dfrac{5}{x}+\dfrac{6}{x^2}}$.

21. $\dfrac{\dfrac{x-5}{2}-7+\dfrac{24}{x}}{\dfrac{9-3x}{x}}$.

22. $\dfrac{\dfrac{1}{x+1}}{1-\dfrac{1}{1+x}}+\dfrac{\dfrac{1}{x+1}}{\dfrac{x}{1-x}}+\dfrac{\dfrac{1}{1-x}}{\dfrac{x}{1+x}}$.

23. $\dfrac{3xyz}{yz+zx+xy}-\dfrac{\dfrac{x-1}{x}+\dfrac{y-1}{y}+\dfrac{z-1}{z}}{\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}}$.

24. $\dfrac{\dfrac{1}{x+y}+\dfrac{2}{x-y}-\dfrac{9}{3x-y}}{\dfrac{-8y}{y^2-9x^2}}$.

25. $\dfrac{\dfrac{x^2+(a+b)x+ab}{x^2-(a+b)x+ab}}{\dfrac{x^2-b^2}{x^2-a^2}}$.

26. $\dfrac{\dfrac{1}{a}+\dfrac{1}{b+c}}{\dfrac{1}{a}-\dfrac{1}{b+c}}\div\dfrac{1}{1+\dfrac{b^2+c^2-a^2}{2bc}}$.

27. $\dfrac{x^2-\dfrac{(x^2+y^2-z^2)^2}{4y^2}}{\dfrac{(x+y)^2-z^2}{y^2}\times\dfrac{(x-y+z)^2}{4}}$.

215. A complex fraction of the form $\dfrac{a}{b+\dfrac{c}{d+\cdots}}$ is called a **continued fraction.**

28. Simplify $\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{x}}}$.

SOLUTION

$$\frac{1}{1+\dfrac{1}{1+\dfrac{1}{x}}}=\frac{1}{1+\dfrac{1}{\dfrac{x+1}{x}}}$$

$$=\frac{1}{1+\dfrac{x}{x+1}}$$

$$=\frac{x+1}{x+1+x}=\frac{x+1}{2x+1}.$$

SUGGESTION. — In the above exercise, the part first simplified is the *last complex part* $\dfrac{1}{1+\dfrac{1}{x}}$, which is reduced to a simple fraction.

Every continued fraction may be simplified by successively reducing its *last complex part* to a simple fraction.

Simplify:

29. $\dfrac{1}{x+\dfrac{1}{1+\dfrac{x+1}{3-x}}}$.

30. $\dfrac{a}{a+1+\dfrac{a}{a+1-\dfrac{1}{a}}}$.

31. $\dfrac{2}{2-\dfrac{2}{2-\dfrac{2}{2-x}}}$.

32. $\dfrac{x-2}{x-2-\dfrac{x}{x-\dfrac{x-1}{x-2}}}$.

33. $\dfrac{1}{a+\dfrac{1}{a+\dfrac{1}{a}}}$.

34. $1+\dfrac{c}{1+c+\dfrac{2c}{1+\dfrac{1}{c}}}$.

REVIEW

16. 1. When is a fraction in its lowest terms?

Define factoring; prime factors; reciprocal of a number.

. When may the factor theorem be used to advantage?

. Define highest common factor; lowest common multiple.

. Give a rule for finding the lowest common multiple of or more expressions.

. Show that the product of $a^2 - b^2$ and $a^3 - b^3$ is equal to product of their highest common factor and lowest common iple.

. Distinguish between an *integral* and a *fractional* alge- expression.

. By what must a fraction be multiplied in order to obtain lowest possible integral expression?

. Why must a broader definition of fraction be given lgebra than in arithmetic? Give the algebraic definition.

. Under what conditions may the sign of the numerator the denominator of a fraction be changed?

.. Show that $\dfrac{5}{9 - x^2} = \dfrac{-5}{x^2 - 9}$.

. Show that the sum of a and b divided by the sum of r reciprocals equals ab.

. Distinguish between a complex fraction and a continued tion.

. Show that dividing $\dfrac{x}{y}$ by x is the same as multiplying y $\dfrac{1}{x}$.

implify:

15. $\dfrac{x^{m+4}}{x^{m-4}}$. 16. $\dfrac{a^{5-2n}}{a^3}$. 17. $-\dfrac{2^{3x+1}}{-2^{1-2x}}$.

ıce to lowest terms:

$\dfrac{x^3+x^2+x-3}{x^3+3x^2+5x+3}$.

20. $\dfrac{x^3+x^2-22x-40}{x^3-7x^2+2x+40}$.

$\dfrac{x^3-x^2-x-2}{x^3+3x^2+3x+2}$.

21. $\dfrac{x^3+10x^2+7x-18}{x^3-8x^2-11x+18}$.

mplify:

22. $\dfrac{x}{2y-1}+\dfrac{y}{2y+1}-\dfrac{y-x}{1-4y^2}$.

23. $\dfrac{a^2}{4(1-a)^2}-\left(\dfrac{3}{8(1-a)}+\dfrac{1}{8(a+1)}-\dfrac{1-a}{4(a+1)}\right)$.

24. $\left(1+\dfrac{2}{m-1}\right)\left(\dfrac{m^2+m-2}{m^2+m}\right)$.

25. $\left(\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}-\dfrac{4ab}{a^2+b^2}\right)\dfrac{a^2-b^2}{8b^2}$.

26. $\left(1+\dfrac{x}{y}\right)\left(\dfrac{x}{y^2}-\dfrac{1}{y}+\dfrac{1}{x}\right)$.

27. $\left(x+1+\dfrac{1}{x}+\dfrac{1}{x^2}\right)\div\left(x+1-\dfrac{1}{x}-\dfrac{1}{x^2}\right)$.

28. $\left(\dfrac{a+b}{a-b}+\dfrac{a^2+b^2}{a^2-b^2}\right)\left(\dfrac{a+b}{a-b}-\dfrac{a^3+b^3}{a^3-b^3}\right)$.

29. $\left(x^2-3xy-2y^2+\dfrac{12y^3}{x+3y}\right)\div\left(3x-6y-\dfrac{2x^2}{x+3}\right.$

30. $\left(\dfrac{m-3n}{m+n}\right)\left(1+\dfrac{4n}{m+n}\right)\div\left(\dfrac{m}{n}+2-\dfrac{15n}{m}\right)$.

31. $1-\dfrac{2x+5x^2}{2(x+1)^2}-\left[\dfrac{x^2+x}{2x-\dfrac{2}{x}}\right]\left(\dfrac{3-3x}{(x+1)^2}\right)$.

32. $\left(1+\dfrac{x}{a-x}\right)\left(\dfrac{x}{x+a}-\dfrac{2x^2+2ax-a^2}{x^2+3ax+2a^2}\right)$.

Expand by inspection:

$$\left(\frac{x}{y}+\frac{y}{x}\right)\left(\frac{x}{y}-\frac{y}{x}\right).$$

$$\left(\frac{x}{y}+\frac{y}{x}\right)\left(\frac{x}{y}+\frac{y}{x}\right).$$

35. $\left(\frac{a}{b}-3\right)\left(\frac{a}{b}+8\right).$

36. $\left(2x-\frac{1}{2x}\right)\left(2x-\frac{1}{2x}\right).$

Simplify:

$$\frac{1+\frac{1}{x^2}+\frac{1}{x^4}}{1+\frac{1}{x}+\frac{1}{x^2}}.$$

$$\frac{\left(\frac{m^2+n^2}{n}-m\right)\div\left(\frac{1}{n}-\frac{1}{m}\right)}{\frac{m^3+n^3}{m^2-n^2}}.$$

$$\frac{\frac{c}{(a+1)^2}+\frac{d}{(a+1)^2}}{\frac{a}{(a+1)^4}+\frac{1}{(a+1)^4}}.$$

$$\frac{1}{1-\frac{1}{1-\frac{1}{1-x}}}.$$

41. $\dfrac{\frac{6a}{x+y}-\frac{4}{x+y}}{\frac{1}{(x+y)^2}}.$

42. $1-\dfrac{\frac{a^2+3a+2}{a^2+2a+1}}{\frac{a^2+7a+12}{a^2+5a+4}}.$

43. $\dfrac{\frac{m^3-n^3}{m^3+n^3}\left(1-\frac{2n}{m+n}\right)}{1+\frac{2mn}{m^2-mn+n^2}}.$

44. $\dfrac{1}{2-\frac{3}{4-\frac{5}{6-x}}}.$

45. $\left(\frac{x+y}{x-y}+\frac{x-y}{x+y}\right)\div\left(\frac{x+y}{2(x-y)}-\frac{x-y}{2x+2y}\right).$

46. $\dfrac{\left(\frac{a}{x^2}+\frac{1}{x}+\frac{1}{a}+\frac{x}{a^2}\right)\left(\frac{a^2}{x^3}-\frac{1}{x}+\frac{x}{a^2}\right)}{\frac{a^3}{x^5}\left(1+\frac{x}{a}\right)}.$

47. $\dfrac{x^3-\frac{8}{y^3}}{x^3y^3-x^2y^2}\times\dfrac{\frac{1}{xy}+\frac{1}{x^2y^2}}{1+\frac{2}{xy}+\frac{4}{x^2y^2}}\times\dfrac{xy-1}{xy+1}.$

SIMPLE EQUATIONS

ONE UNKNOWN NUMBER

217. The student has already learned what an equation is (§ 4), and he has solved many equations and problems. In this chapter and the next, however, he will find a more complete and comprehensive treatment of the subject, extended to some kinds of equations that are new to him.

218. An equation all of whose known numbers are expressed by figures is called a **numerical equation.**

219. An equation one or more of whose known numbers is expressed by letters is called a **literal equation.**

220. An equation that does not involve an unknown number in any denominator is called an **integral equation.**

$x + 5 = 8$ and $\frac{2x}{3} + 5 = 8$ are integral equations. Though the second equation contains a fraction, the unknown number x does not appear in the denominator.

221. An equation that involves an unknown number in any denominator is called a **fractional equation.**

$x + 5 = \frac{8}{x}$ and $\frac{2x}{x-1} = 7$ are fractional equations.

222. An equation whose members are identical, or such that they may be reduced to the same form, is called an **identical equation**, or an **identity.**

$a + b = a + b$ and $a^2 - b^2 = (a + b)(a - b)$ are identical equations.

An equation whose members are numerical is evidently an *identical equation.*

$10 = 6 + 4$ and $8 \times 2 = 6 + 12 - 2$ are identical equations.

iteral equation that is true for all values of the letters ved is an identical equation, or an identity.

$+ y)^2 = x^2 + 2xy + y^2$ **is an identity, because it is true for all values ad y.**

. An equation that is true for only certain values of its rs is called an **equation of condition.**

ı equation of condition is usually termed simply an **equa-**

$+ 4 = 10$ is an equation of condition, because it is true only when the of x is 6. $x^2 = 9$ is an equation of condition, because it is true only the value of x is $+ 3$ or $- 3$.

4. When an equation is reduced to an identity by the titution of certain known numbers for the unknown num- the equation is said to be **satisfied.**

hen $x = 2$, the equation $3x + 4 = 10$ becomes $6 + 4 = 10$, an identity; quently, the equation is satisfied.

5. Any number that satisfies an equation is called a **root** e equation.

s a root of the equation $3x + 4 = 10$.

6. Finding the roots of an equation is called **solving the ion.**

7. An integral equation that involves only the first power ıe unknown number in any term when the similar terms been united is called a **simple equation,** or an **equation of irst degree.**

$: + 4 = 10$ and $x + 2y - z = 8$ are simple equations.

ır reasons that will be apparent later on, simple equations ıometimes called **linear equations.**

8. Two equations that have the same roots, each equation ng all the roots of the other, are called **equivalent equations.**

$+ 3 = 7$ and $2x = 8$ *are equivalent* equations, each being satisfied for *and for no other value of x.*

229. By the axioms in § 68, if the members of an equation are equally increased or diminished or are multiplied or divided by the same or equal numbers, the two resulting numbers are *equal* and form an equation. But it does not necessarily follow that the equation so formed is *equivalent* to the given equation.

For example, if both members of the equation $x+2=5$, whose only root is $x=3$, are multiplied by $x-1$, the resulting numbers, $(x+2)(x-1)$ and $5(x-1)$, are *equal* and form an equation,

$$(x+2)(x-1)=5(x-1),$$

which is not *equivalent* to the given equation, since it is satisfied by $x=1$ as well as by $x=3$; that is, the root $x=1$ has been *introduced.*

In applying axioms to the solution of equations we endeavor to change to *equivalent* equations, each simpler than the preceding, until an equation is obtained having the unknown number in one member and the known numbers in the other.

The following principles serve to guard the student against *introducing* or *removing* roots without accounting for them:

230. Principles. — **1.** *If the same expression is added to or subtracted from both members of an equation, the resulting equation is equivalent to the given equation.*

2. *If both members of an equation are multiplied or divided by the same known number, except zero, the resulting equation is equivalent to the given equation.*

3. *If both members of an integral equation are multiplied by the same unknown integral expression, the resulting equation has all the roots of the given equation and also the roots of the equation formed by placing the multiplier equal to zero.*

It follows from Prin. 3 that it is *not allowable to remove from both members of an equation a factor that involves the unknown number, unless the factor is placed equal to zero and the root of this equation preserved.*

Thus, if $x-2$ is removed from both members of the equation $(x-2)(x+4)=7(x-2)$, the resulting equation $x+4=7$ has only *the root* $x=3$*;* consequently, the root of $x-2=0$, *removed* by dividing *by the factor* $x-2$, should be preserved.

Clearing Equations of Fractions

.. The process of changing an equation containing frac- to an equation without fractions is called **clearing the ion of fractions.**

EXERCISES

‡. 1. Solve the equation $\frac{x-8}{2}=6-\frac{x}{3}$.

PROCESS

$$\frac{x-8}{2}=6-\frac{x}{3}$$

$$-24=36-2x$$

$$5x=60$$

$$x=12$$

EXPLANATION. — Since the first fraction will become an integer if the members of the equation are multiplied by 2 or a multiple of 2, and since the second fraction will become an integer if the members of the equation are multiplied by 3 or a multiple of 3, the equation may be cleared of fractions in a single operation by multiplying both members by some *common* multiple of 2 and 3, or 12, or 18, etc. It is usually best to use the L.C.M. of the denomis, which in this case is 6.

en, multiplying both members by 6, transposing terms, and dividing e coefficient of x, we obtain $x = 12$.

ERIFICATION. — When 12 is substituted for x, the given equation bes $2 = 2$, an identity; consequently, § 224, the equation is satisfied $= 12$.

Solve the equation $\frac{x-1}{2}-\frac{x-2}{3}=\frac{2}{3}-\frac{x-3}{4}$.

SOLUTION

$$\frac{x-1}{2}-\frac{x-2}{3}=\frac{2}{3}-\frac{x-3}{4}.$$

ultiplying both members of the equation by the L.C.M. of the deinators, which in this case is 12, we obtain

$$6(x-1)-4(x-2)=8-3(x-3).$$

xpanding, $\quad 6x-6-4x+8=8-3x+9.$

ransposing, etc., $\quad 5x=15.$

ence, $\quad x=3.$

ERIFICATION. — *When* $x = 3$, the given equation becomes $\frac{1}{3}=\frac{1}{3}$, an *ity; consequently, the* equation is satisfied by $x = 3$.

To clear equations of fractions:

RULE. — *Multiply both members of the equation by the lowest common multiple of the denominators.*

1. To simplify the work and to avoid introducing roots, reduce all fractions to their lowest terms and unite fractions that have like denominators before clearing.

2. If a fraction is negative, the signs of all the terms of the numerator must be changed when the denominator is removed.

3. Roots are sometimes introduced in clearing of fractions. Such roots may be discovered by verification. Those which do not satisfy the given equation are not roots of it, and should be rejected.

Solve, and verify each result:

3. $2x + \frac{x}{3} = \frac{35}{3}.$

4. $\frac{x}{4} + 10 = 13.$

5. $\frac{x}{6} + 2x = 26.$

6. $3x - \frac{x}{5} = 14.$

7. $\frac{x}{2} + \frac{x}{6} = \frac{10}{3}.$

8. $7\frac{1}{2} - \frac{3x}{14} = \frac{x}{7}.$

9. $\frac{x}{5} + \frac{x}{7} = 24.$

10. $\frac{2x}{3} - \frac{5}{6} = \frac{x}{4}.$

11. $\frac{x}{2} + \frac{x}{3} - \frac{x}{4} + \frac{3x}{10} - \frac{5x}{12} = 7.$

12. $\frac{25x}{18} - \frac{5x}{9} + \frac{2x}{3} - \frac{5x}{6} = 2.$

13. $\frac{2x}{3} - \frac{7x}{8} + \frac{5x}{18} + \frac{x}{24} = \frac{4}{9}.$

14. $\frac{3x}{4} + \frac{7x}{16} - \frac{x}{2} - \frac{9x}{16} = \frac{1}{8}.$

15. $\frac{15x}{7} + \frac{5x}{6} - \frac{11x}{3} + \frac{19x}{14} = 2.$

16. $\frac{2x}{15} + \frac{5x}{25} - \frac{4x}{9} + \frac{x}{6} = \frac{1}{9}.$

17. $\frac{3x}{4}-\frac{7x}{12}=\frac{11x}{36}-\frac{8x}{9}+\frac{3}{2}.$

18. $\frac{y-1}{2}+\frac{y-2}{3}+\frac{y-3}{4}=\frac{5y-1}{6}.$

19. $\frac{v+1}{3}-\frac{v+4}{5}+\frac{v+3}{4}=16.$

20. $\frac{7z+2}{6}-\frac{12-z}{4}+\frac{z+2}{2}=6.$

21. $\frac{u-3}{7}+\frac{u+5}{3}-\frac{u+2}{6}=4.$

22. $\frac{3t-5}{4}-\frac{7t-13}{6}=3-\frac{t+3}{2}.$

23. $\frac{5x+2}{3}-\left(x-\frac{3x-1}{2}\right)=\frac{3x+19}{2}-\left(\frac{x+1}{6}+5\right).$

24. $1.07x+.32=.15x+10.12+.675x.$

GGESTION. — Clear of decimal fractions by multiplying by 1000.

25. $.604x-3.16-.7854x+7.695=0.$

26. $3.1416x-15.5625+.0216x=.2535.$

27. $\frac{.2x}{7}-\frac{.1x}{4}-\frac{.1x}{2}+\frac{.4x}{7}=\frac{.3}{14}.$

28. $\frac{n+4}{.3}+\frac{2-2n}{.6}=\frac{n+1}{.2}-\frac{10}{.3}.$

29. $\frac{9x+5}{14}+\frac{8x-7}{6x+2}=\frac{36x+15}{56}+\frac{10\frac{1}{4}}{14}.$

JGGESTION. — The equation may be written

$$\frac{9x}{14}+\frac{5}{14}+\frac{8x-7}{6x+2}=\frac{36x}{56}+\frac{15}{56}+\frac{41}{56}.$$

30. $\dfrac{3x-2}{2x-5}+\dfrac{3x-21}{5}=\dfrac{6x-22}{10}.$

31. $\dfrac{4x+3}{9}=\dfrac{8x+19}{18}-\dfrac{7x-29}{5x-12}.$

32. $\dfrac{6p+1}{15}-\dfrac{2p-4}{7p-13}=\dfrac{2p-1}{5}.$

33. $\dfrac{10q+17}{18}-\dfrac{5q-2}{9}=\dfrac{12q-1}{11q-8}.$

34. $\dfrac{6r+3}{15}-\dfrac{3r-1}{5r-25}=\dfrac{2r-9}{5}.$

35. Solve the equation $\dfrac{x-1}{x-2}+\dfrac{x-6}{x-7}=\dfrac{x-5}{x-6}+\dfrac{x-2}{x-3}.$

Solution. — It will be observed that if the fractions in each member were connected by the sign −, and if the terms of each member were united, the numerators of the resulting fractions would be simple. The fractions can be made to meet this condition by transposing one fraction in each member.

Consequently, it is sometimes expedient to defer clearing of fractions.

Transposing, $\dfrac{x-1}{x-2}-\dfrac{x-2}{x-3}=\dfrac{x-5}{x-6}-\dfrac{x-6}{x-7}.$

Uniting terms, $\dfrac{-1}{x^2-5x+6}=\dfrac{-1}{x^2-13x+42}.$

Since the fractions are equal and their numerators are equal, their denominators must be equal.

Then, $x^2-5x+6=x^2-13x+42.$

$\therefore x=4\frac{1}{2}.$

36. $\dfrac{x-1}{x-2}+\dfrac{x-7}{x-8}=\dfrac{x-5}{x-6}+\dfrac{x-3}{x-4}.$

37. $\dfrac{x-3}{x-4}+\dfrac{x-7}{x-8}=\dfrac{x-6}{x-7}+\dfrac{x-4}{x-5}.$

38. $\dfrac{v+2}{v+1}-\dfrac{v+3}{v+2}=\dfrac{v+5}{v+4}-\dfrac{v+6}{v+5}.$

39. $\dfrac{s+1}{s+2}+\dfrac{s+6}{s+7}=\dfrac{s+2}{s+3}+\dfrac{s+5}{s+6}.$

40. $\dfrac{x^3+1}{x-1}-\dfrac{x^3-1}{x+1}=\dfrac{8}{x^2-1}+2\,x.$

41. $\dfrac{x^3+2}{x+1}-\dfrac{x^3-2}{x-1}=\dfrac{10}{x^2-1}-2\,x.$

42. $\dfrac{r}{2}(2-r)-\dfrac{r}{4}(3-2\,r)=\dfrac{r+10}{6}.$

43. $\dfrac{3\,n-4}{4}-\left(\dfrac{4\,n}{5}+\dfrac{n+2}{2}\right)=\dfrac{9\,n}{10}-\left(19+\dfrac{n+4}{2}\right).$

44. $\dfrac{(x-3)^2}{7}-\dfrac{(x+4)^2}{3}=20-\left(8\,x+\dfrac{5\,x+10}{21}\right)-\dfrac{4\,x^2}{21}.$

45. $\dfrac{\dfrac{2\,c}{3}+4}{2}=\dfrac{7\frac{1}{2}-c}{3}+\dfrac{c}{2}\left(\dfrac{6}{c}-1\right).$

46. $\dfrac{\dfrac{4}{5\,d}-16}{24}-\dfrac{\dfrac{2}{5\,d}+6}{60}=\dfrac{4\frac{1}{6}}{5}.$

47. $\dfrac{2\,x\left(1-\dfrac{5}{x}\right)}{3}+\dfrac{3\,x\left(1-\dfrac{4}{x}\right)}{4}=\dfrac{x-4}{\frac{4}{5}}.$

48. $\dfrac{1}{2}\,x-2\left(\dfrac{4\,x}{5}-3\right)=4-\dfrac{3}{2}\left(\dfrac{x}{2}+1\right).$

49. $\dfrac{(2\,x+1)^2}{.05}-\dfrac{(4\,x-1)^2}{.2}=\dfrac{15}{.08}+\dfrac{3(4\,x+1)}{.4}.$

50. $\dfrac{17+\dfrac{3}{x}}{3}+\dfrac{1+\dfrac{18}{x}}{5}=\dfrac{\dfrac{21}{x}-1}{9}+\dfrac{\dfrac{100}{x}+\dfrac{5}{3}}{15}.$

51. $\dfrac{\frac{1}{4}(x-4)}{\frac{3}{2}}-\dfrac{4\,x-16}{6}=\dfrac{3}{5}-\dfrac{\dfrac{2\,x}{5}+5}{\frac{5}{2}}.$

Literal Equations

233. 1. Solve the equation $\frac{x-b^2}{a}=\frac{x-a^2}{b}$ for x.

SOLUTION

$$\frac{x-b^2}{a}=\frac{x-a^2}{b}.$$

Clearing of fractions, $bx - b^3 = ax - a^3.$

Transposing, etc., $ax - bx = a^3 - b^3.$

$(a-b)x = a^3 - b^3.$

Dividing by $(a-b)$, $x = a^2 + ab + b^2.$

VERIFICATION. — When $a = 2$ and $b = 1$, $x = a^2 + ab + b^2 = 4 + 2 + 1 = 7$, and the given equation becomes $\frac{7-1}{2}=\frac{7-4}{1}$, or $3 = 3$, an identity; consequently, the equation is satisfied by $x = a^2 + ab + b^2$.

Solve for x, and verify each result:

2. $\frac{c^2-x}{nx}+\frac{n^2}{cx}=\frac{1}{c}.$

3. $1-\frac{ab}{x}=\frac{7}{ab}-\frac{49}{abx}.$

4. $\frac{a^3}{ab^2}-\frac{2a^2}{b^2x}=1-\frac{2b^2}{a^2x}.$

5. $\frac{x}{b}-\frac{x+2b}{a}=\frac{a}{b}-3.$

6. $\frac{x-2ab}{cx}-\frac{1}{x}=\frac{x-3c}{abx}.$

7. $\frac{x-a}{b}+\frac{2x}{a}=5+\frac{6b}{a}.$

8. $\frac{a^2}{bx}+\frac{b^2}{ax}=\frac{a+b}{ab}-\frac{3(a+b)}{x}.$

9. $\frac{a^2+b^2}{2bx}-\frac{a-b}{2bx^2}=\frac{b}{x}.$

10. $\frac{2x-a}{x-a}-\frac{x-a}{x+a}=1.$

11. $\frac{x-2a}{a}+\frac{x}{b}=\frac{a^2+b^2}{ab}.$

12. $6x+18\left(1-\frac{a}{2}\right)=a(x-a).$

13. $b(2x-9c-14b)=c(c-x).$

14. $a(x-a-2b)+b(x-b)+c(x+c)=0.$

15. $(a-x)(x-b)+(a+x)(x-b)=(a-b)^2.$

16. $(a-b)(x-c)-(b-c)(x-a)=(c-a)(x-b).$

17. $\dfrac{a-b+c}{x+a}=\dfrac{b-a+c}{x-a}.$

18. $\dfrac{1}{a(b-x)}+\dfrac{1}{b(c-x)}-\dfrac{1}{a(c-x)}=0.$

19. $\dfrac{x-1}{a-1}-\dfrac{a-1}{x-1}=\dfrac{x^2-a^2}{(a-1)(x-1)}.$

20. $\dfrac{1}{m+n}-\dfrac{2\,mn}{(m+n)^3}-\dfrac{m}{(m+n)^2}=\dfrac{x-n}{(m+n)^2}.$

21. $\dfrac{x+a}{b}+\dfrac{x+c}{a}+\dfrac{x+b}{c}=\dfrac{a}{b}+\dfrac{b}{c}+\dfrac{c}{a}+1.$

22. $\dfrac{x}{a+b+c}+\dfrac{x}{a+b-c}=a^2+b^2+c^2+2\,ab.$

23. $\dfrac{a+x}{a}-\dfrac{2\,x}{a+x}+\dfrac{x^2(x-a)}{a(a^2-x^2)}=\dfrac{1}{3}.$

SUGGESTION — Simplify as much as possible before clearing of fractions.

24. $\dfrac{x^2-ax-bx+ab}{x-a}=\dfrac{x^2-2\,bx+2\,b^2}{x-b}-\dfrac{c^2}{x-c}.$

Algebraic Representation

234. 1. What part of $m-n$ is p?

2. From what number must m be subtracted to produce n?

3. How much less is $\dfrac{m}{n}$ dollars than m dollars?

4. Indicate the sum of l and m divided by 2, and that result multiplied by n.

5. Indicate the product of s and $(r-1)$ divided by the nth power of the sum of t and v.

6. A boy who had m marbles lost $\dfrac{1}{a}$ of them. How many marbles had he left?

7. By what number must x be multiplied that the product shall be z?

8. Indicate the expression for $\frac{1}{2}$ the product of g and the square of t.

9. Indicate the square of x, plus twice the product of x and y, plus the square of y, divided by the sum of x and y.

10. By what number must the sum of x and $-y$ be multiplied to produce the square of x minus the square of y?

11. Indicate the result when the sum of a, b, and $-c$ is to be divided by the square of the sum of a and b.

12. It is t miles from Albany to Utica. The Empire State Express runs s miles an hour. How long does it take this train to go from Albany to Utica?

13. A cabinetmaker worked x days on two pieces of work. For one he received v dollars, and for the other w dollars. What were his average earnings per day for that time?

14. A train runs x miles an hour and an automobile $x-y$ miles an hour. How much longer will it take the automobile to run s miles than the train?

15. Indicate the result when b is added to the numerator and subtracted from the denominator of the fraction $\frac{a}{c}$.

16. A farmer had $\frac{1}{x}$ of his crop in one field, $\frac{1}{y}$ in a second, and $\frac{1}{z}$ in a third. What part of his crop had he in these three fields?

17. A won m more games of tennis than B, and B won n more games than C. How many more games did A win than C?

18. A student spends $\frac{1}{m}$ of his income for room rent, $\frac{1}{n}$ for board, $\frac{1}{s}$ for books, and $\frac{1}{r}$ for clothing. If his income is x dollars, how much has he left?

Problems

235. Review the general directions for solving problems ven in § 77.

1. A grocer paid $ 8.50 for a molasses pump and 5 feet of bing. He paid 12 times as much for the pump as for each ot of tubing. How much did the pump cost? the tubing?

SUGGESTION. — If we knew the cost of a foot of tubing, we could compte the cost of the pump. Therefore, let x represent the number of cents e foot of tubing cost.

2. A merchant purchased an assortment of bath robes for 180. By selling $\frac{1}{4}$ of them at $ 6 each, $\frac{1}{6}$ of them at $ 7 each, of them at $ 5 each, and $\frac{1}{4}$ of them at $ 8 each, he gained 128. How many bath robes did he sell at each price?

3. A shipment of 12,000 tons of coal arrived at Boston on 3 rges and 2 schooners. Each schooner held $3\frac{1}{2}$ times as much each barge. Find the capacity of a barge; of a schooner.

4. A salmon cannery in Alaska paid 2¢ for each red fish ught and 10¢ for each king salmon. Two men brought in 0 fish one day and received $ 24.80 for their catch. How ny fish of each kind did they catch?

5. The powder and the shell used in a twelve-inch gun igh 1265 pounds. The powder weighs 15 pounds more than s much as the shell. Find the weight of each.

6. A merchant bought 62 barrels of flour, part at $ $4\frac{3}{4}$ per rel, the rest at $ $5\frac{1}{2}$ per barrel. If he paid $ 320 for the ır, how many barrels of each grade did he buy?

7. During a year of 365 days one locality had 6 days less "clear" weather than of "cloudy" weather, and 4 days re of "clear" than of "partly cloudy" weather. Find the mber of days of each kind of weather during the year.

8. The bark from a cork tree lost $\frac{1}{5}$ of its weight by being led. The boiled cork was then scraped, its weight thus ng reduced $\frac{1}{4}$. *How much* did the cork weigh before and *r these two operations,* if the entire loss was 16 pounds?

9. An acre of wheat yielded 2000 pounds more of straw than of grain. The weight of the grain was .3 of the total weight. How many 60-pound bushels of wheat were produced?

10. The precious stones imported into this country one year were valued at \$40,000,000. The uncut diamonds were worth $\frac{4}{5}$ as much as the cut diamonds but twice as much as the other precious stones. Find the value of each kind of diamonds.

11. The cost per mile of running a train was 14¢ less with electrical equipment than with steam, or $\frac{3}{5}$ as much. What was the cost per mile with electricity?

12. A Chicago merchant paid \$43.75 to keep 7000 pounds of butter in storage for 4 months. For each of the last 3 months he was charged $\frac{1}{2}$ as much as for the first month. What was the charge per pound for the first month, and for each succeeding month?

13. A newspaper reporter saved $\frac{1}{5}$ of his weekly salary, or \$1 more than was saved by an artist on the same paper, whose salary was \$5 greater but who saved only $\frac{1}{7}$ of it. How much did the reporter earn per week? the artist?

14. A field is twice as long as it is wide. By increasing its length 20 rods and its width 30 rods, the area will be increased 2200 square rods. What are its dimensions?

15. In a purse containing \$1.45 there are $\frac{1}{2}$ as many quarters as 5-cent pieces and $\frac{2}{3}$ as many dimes as 5-cent pieces. How many pieces are there of each kind?

16. Find a fraction whose value is $\frac{4}{7}$ and whose denominator is 15 greater than its numerator.

17. Find a fraction whose value is $\frac{2}{3}$ and whose numerator is 3 greater than half of its denominator.

18. The numerator of a certain fraction is 8 less than the denominator. If each term of the fraction is decreased by 5, the resulting fraction equals $\frac{1}{3}$. What is the fraction?

19. An experienced woman reeled .12 Kg. more silk per day from cocoons than a beginner. During one week (6 days) she had $\frac{3}{4}$ of a day lost time, yet she reeled .36 Kg. more than the beginner. How much did each reel per day?

20. A can do a piece of work in 8 days. If B can do it in 10 days, in how many days can both working together do it?

Solution

Let $x =$ the required number of days.

Then, $\frac{1}{x} =$ the part of the work both can do in 1 day,

$\frac{1}{8} =$ the part of the work A can do in 1 day,

$\frac{1}{10} =$ the part of the work B can do in 1 day;

$$\therefore \frac{1}{x} = \frac{1}{8} + \frac{1}{10}, \text{ or } \frac{9}{40}.$$

Solving, $x = \frac{40}{9}$, or $4\frac{4}{9}$, the required number of days.

21. A can do a piece of work in 10 days, B in 12 days, and C in 8 days. In how many days can all together do it?

22. A can pave a walk in 6 days, and B can do it in 8 days. How long will it take A to finish the job after both have worked 3 days?

23. A can build a wall in 15 days, but with the aid of B and C the wall can be built in 6 days. If B does as much work in 1 day as C does in 2 days, in how many days can B and C separately build the wall?

24. A and B can dig a ditch in 10 days, B and C can dig it in 6 days, and A and C in $7\frac{1}{2}$ days. In what time can each man do the work?

Suggestion. —Since A and B can dig $\frac{1}{10}$ of the ditch in 1 day, B and C $\frac{1}{6}$ of it in 1 day, and A and C $\frac{2}{15}$ of it in 1 day, $\frac{1}{10} + \frac{1}{6} + \frac{2}{15}$ is twice the part they can all dig in 1 day.

25. A and B can load a car in 3 hours, B and C in $2\frac{1}{2}$ hours, and A and C in $2\frac{1}{7}$ hours. How long will it take each alone to load it?

26. The units' digit of a number expressed by two digits exceeds the tens' digit by 5. If the number increased by 63 is divided by the sum of its digits, the quotient is 10. Find the number.

SOLUTION

Let $x =$ the digit in tens' place.

Then, $x + 5 =$ the digit in units' place,

and $10\,x + (x + 5) =$ the number;

$$\therefore \quad \frac{10\,x + (x+5) + 63}{2\,x + 5} = 10;$$

whence, $x = 2$,

and $x + 5 = 7$.

Therefore, the number is 27.

27. The tens' digit of a number expressed by two digits is 3 times the units' digit. If the number diminished by 33 is divided by the difference of the digits, the quotient is 10. What is the number?

28. The tens' digit of a number expressed by two digits is $\frac{1}{2}$ of the units' digit. If the number increased by 27 is divided by the sum of its digits, the quotient is $6\frac{1}{4}$. What is the number?

29. A girl found that she could buy 18 apples with her money and have 5 cents left, or 12 oranges and have 11 cents left, or 8 apples and 6 oranges and have 10 cents left. How much money had she?

30. An officer, attempting to arrange his men in a solid square, found that with a certain number of men on a side he had 34 men over, but with one man more on a side he needed 35 men to complete the square. How many men had he?

SUGGESTION. — With x men on a side, the square contained x^2 men; with $x + 1$ men on a side, there were places for $(x + 1)^2$ men.

31. A regiment drawn up in the form of a solid square was reënforced by 240 men. When the regiment was formed again in a solid square, there were 4 more men on a side. How many men were there in the regiment at first?

2. Mr. Reynolds invested $ 800, a part at 6 %, the rest at . The total annual interest was $ 45. Find how much ley he invested at each rate.

UGGESTION. — Let x = the number of dollars invested at 6 %.

hen, $800 - x$ = the number of dollars invested at 5 %;

$$\therefore \tfrac{6}{100}x + \tfrac{5}{100}(800 - x) = 45.$$

3. A man has $\frac{3}{5}$ of his property invested at 4 %, $\frac{1}{4}$ at 3 %, the remainder at 2 %. How much is his property valued if his annual income is $ 860?

4. A man put out $ 4330 in two investments. On one of m he gained 12 %, and on the other he lost 5 %. If his net 1 was $ 251, what was the amount of each investment?

5. Mr. Bailey loaned some money at 4 % interest, but re-'ed $ 48 less interest on it annually than Mr. Day, who had led $\frac{4}{5}$ as much at 6 %. How much did each man loan?

6. A man paid $ 80 for insuring two houses for $ 6000 and)00, respectively. The rate for the second house was $\frac{1}{8}$ % iter than that for the first. What were the two rates?

7. Mr. Barnes received a yearly income of 7 % from an 'stment. He borrowed twice as much as he already had 'sted, paying 5 % interest, put this sum with his original in-ment, and then received a net income of $ 385. What was sum first invested and the sum borrowed?

3. A man bought some 50-dollar shares in one stock com-y and $\frac{3}{5}$ as many 100-dollar shares in another. At the end ie first quarter, dividends of 2 % and of $1\frac{1}{2}$ %, respectively, declared on these stocks, and the man received $ 120. How h money did he invest in each company?

). My deposit in a savings bank that pays 4 % interest times as great as my deposit with a trust company. On the r I receive no interest, but would receive 3 % interest if the)sit were $ 300 or more. If I transfer $ 400 to the trust pany, my interest income for the next quarter will be in-sed $ $\frac{1}{2}$. Find my present deposit in each place.

40. At what time between 5 and 6 o'clock will the hands of a clock be together?

SOLUTION

Starting with the hands in the position shown, at 5 o'clock, let x represent the number of minute spaces passed over by the minute hand after 5 o'clock until the hands come together. In the same time the hour hand will pass over $\frac{1}{12}$ of x minute spaces.

Since they are 25 minute spaces apart at 5 o'clock,

$$x - \frac{x}{12} = 25;$$

$\therefore x = 27\frac{3}{11}$, the number of minutes after 5 o'clock.

41. At what time between 1 and 2 o'clock will the hands of a clock be together?

42. At what time between 6 and 7 o'clock will the hands of a clock be together?

43. At what time between 10 and 11 o'clock will the hands of a clock point in opposite directions?

44. At what two different times between 4 and 5 o'clock will the hands of a clock be 15 minute spaces apart?

45. A man rows downstream at the rate of 6 miles an hour and returns at the rate of 3 miles an hour. How far downstream can he go and return within 9 hours?

46. A motor boat went up the river and back in 2 hours 56 minutes. Its rate per hour was $17\frac{1}{2}$ miles going up and 21 miles returning. How far up the river did it go?

47. An express train whose rate is 40 miles an hour starts 1 hour and 4 minutes after a freight train and overtakes it in 1 hour and 36 minutes. How many miles does the freight train run per hour?

48. A yacht goes 5 miles downstream in the same time that it goes 3 miles upstream; but if its rate each way is diminished 4 miles an hour, its rate downstream will be twice its rate upstream. How fast does it go in each direction?

9. The distance by canal from Albany to Syracuse is 166 es. A canal boat leaves Albany for Syracuse, moving at rate of 3 miles in 2 hours; at the same time another leaves racuse for Albany, moving at the rate of 5 miles in 4 hours. ow far from Albany do they meet?

50. The distance between Southampton and New York is 46 nautical miles, or knots. A vessel left Southampton r New York and sailed at the rate of 20 knots an hour. $15\frac{1}{2}$ ours later another vessel started from New York and sailed ver the same route at the rate of 18 knots an hour. How far rom Southampton were the vessels when they met?

51. At 3 P.M. on Monday some people started by boat from oronto for Montreal, where they remained 36 hours. They eturned by rail, reaching Toronto at 4:30 P.M., Thursday. he average rate was 13 miles per hour by boat and 44 miles er hour by rail. The distance was 60 miles greater by boat han by rail. What were the distance and the time each way?

52. It took a passenger train, 175 feet long, $7\frac{1}{2}$ seconds to ass completely a freight train, 485 feet long, moving in the pposite direction. If the passenger train was going 3 times as ast as the freight train, find the rate of each per hour.

53. In making 5000 pounds of brass there were used $8\frac{1}{2}$ imes as much copper as tin, and twice as much tin as zinc. ow many pounds of each metal were used?

54. In a quantity of gunpowder the niter composed 10 ounds more than $\frac{2}{3}$ of the weight, the sulphur 3 pounds more han $\frac{1}{12}$ of it, and the charcoal 3 pounds less than $\frac{1}{10}$ of the eight of the niter. What was the weight of the gunpowder?

55. United States silver coins are $\frac{9}{10}$ pure silver, or '$\frac{9}{10}$ fine.' ow much pure silver must be melted with 250 ounces of ilver $\frac{4}{5}$ fine to render it of the standard fineness for coinage?

56. In an alloy of 90 ounces of silver and copper there are ounces of silver. How much copper must be added that 10 unces of the new alloy may contain $\frac{2}{5}$ of an ounce of silver?

57. If 80 pounds of sea water contain 4 pounds of salt, how much fresh water must be added that 49 pounds of the new solution may contain $1\frac{3}{4}$ pounds of salt?

58. In an alloy of 75 pounds of tin and copper there are 12 pounds of tin. How much copper must be added that the new alloy may be $12\frac{1}{2}\%$ tin?

59. If in 60 pounds of a solution of salt and water there are 3 pounds of salt, how much fresh water must be evaporated from the solution that 25 pounds of the new solution shall contain $2\frac{1}{2}$ pounds of salt?

60. Of 24 pounds of salt water, 12 % is salt. In order to have a solution that shall contain 4 % salt, how many pounds of pure water should be added?

61. It is desired to add sufficient water to 6 gallons of alcohol 95 % pure to make a mixture 75 % pure. How many gallons of water are required?

62. Four gallons of alcohol 90 % pure is to be made 50 % pure. What quantity of water must be added?

63. A body placed in a liquid loses as much weight as the weight of the liquid displaced. A piece of glass having a volume of 50 cubic centimeters weighed 94 grams in air and 51.6 grams in alcohol. How many grams did the alcohol weigh per cubic centimeter?

64. Brass is $8\frac{2}{5}$ times as heavy as water, and iron $7\frac{1}{2}$ times as heavy as water. A mixed mass weighs 57 pounds, and when immersed displaces 7 pounds of water. How many pounds of each metal does the mass contain?

SUGGESTION. — Let there be x pounds of brass and $(57 - x)$ pounds of iron. Then, x pounds of brass will displace $(x \div 8\frac{2}{5})$ pounds of water.

65. If 1 pound of lead loses $\frac{2}{23}$ of a pound, and 1 pound of iron loses $\frac{2}{15}$ of a pound when weighed in water, how many pounds of lead and of iron are there in a mass of lead and iron that weighs 159 pounds in air and 143 pounds in water?

66. If tin and lead lose, respectively, $\frac{5}{37}$ and $\frac{2}{23}$ of their weights when weighed in water, and a 60-pound mass of lead and tin loses 7 pounds when weighed in water, find the weight of the tin in this mass.

67. If 97 ounces of gold weighs 92 ounces when it is weighed in water, and 21 ounces of silver weighs 19 ounces when it is weighed in water, how many ounces of gold and of silver are there in a mass of gold and silver that weighs 320 ounces in air and 298 ounces in water?

68. If zinc weighs 437.5 pounds per cubic foot and copper 550 pounds, what per cent by volume is each of these metals in an alloy of them, 1 cubic foot of which weighs 532 pounds?

Solution of Formulæ

236. A formula expresses a principle or a rule in symbols. The solution of problems in commercial life, and in mensuration, mechanics, heat, light, sound, electricity, etc., often depends upon the ability to solve formulæ.

EXERCISES

237. 1. The circumference of a circle is equal to π ($=3.1416$) times the diameter, or

$$C = \pi D.$$

Solve the formula for D and find, to the nearest inch, the diameter of a locomotive wheel whose circumference is 194.78 inches.

SOLUTION

From $C = \pi D$, $\qquad \pi D = C.$

$$\therefore\ D = \frac{C}{\pi} = \frac{194.78}{3.1416} = 62.0+.$$

Hence, to the nearest inch, the diameter is 62 inches.

2. Area of a triangle $= \frac{1}{2}$ (base $\times$ altitude), or

$$A = \frac{1}{2}\,bh.$$

Solve for b, then find the base of a triangle whose area is 600 square feet and altitude 40 feet.

3. The area of a trapezoid is equal to the product of half the sum of the bases and the altitude; that is,

$$A = \frac{b + b'}{2} \cdot h.$$

The bases are b and b'; b' is read 'b-prime.'

Solve for h, then find the altitude of a trapezoid whose area is 96 square inches and whose bases are 14 inches and 10 inches, respectively.

4. The volume of a pyramid = $\frac{1}{3}$ (base × altitude), or

$$V = \frac{1}{3} Bh.$$

Solve for B, then find the area of the base of a pyramid whose volume is 252 cubic feet and altitude 9 feet.

5. The charge (c) for a telegram from New York to Chicago, 40¢ for 10 words and 3¢ for each additional word, may be found by the formula,

$$c = 40 + 3\,(n - 10),$$

in which n stands for the number of words.

Find the cost of a 16-word message.

Solve for n, then find how many words can be sent for \$1.

6. In the formula $\quad i = p \cdot \dfrac{r}{100} \cdot t,$

i denotes the interest on a principal of p dollars at simple interest at r % for t years.

Solve for t, then find the time \$300 must be on interest at 5% to yield \$60 interest.

Solve for r. At what rate of interest will \$4500 yield \$900 interest in 5 years?

Solve for p. What principal at $3\frac{1}{2}$% will yield \$210 annually?

7. In § 42 was given the formula for the space (s) passed over by anything that moves with uniform velocity (v) during a given time (t). It is

$$s = vt.$$

Solve for v, then find the velocity of sound when the conditions are such that it travels 8640 feet in 8 seconds.

8. The formula for converting a temperature of F degrees Fahrenheit into its equivalent temperature of C degrees Centigrade is

$$C = \frac{5}{9}(F - 32).$$

Solve for F and express 25° Centigrade (the mean annual temperature in Havana) in degrees Fahrenheit.

9. If a steel rail at 0° C. is heated, for every degree it is heated it will expand a certain part of its original length. Let L_0 (read, 'L sub-zero') denote its original length at 0° C., L its length at t degrees C., and a the certain fractional multiplier, or 'coefficient of expansion.'

Then, $$L = L_0 + L_0 \cdot at.$$

Solve for a. A steel rail 30 feet long at 0° C. expanded to a length of 30.001632 feet at 50° C. Find the value of a.

Solve:

10. $s = \frac{1}{2}at^2$, for a.

11. $F = Ma$, for a.

12. $F = \mu W$, for μ.

13. $W = Fs$, for s.

14. $P = I^2R$, for R.

15. $Mv_1 = mv_2$, for m.

16. $E = \frac{1}{2}Mv^2$, for M.

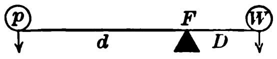
17. $P_0V_0 = PV$, for P.

18. $s = v_0t + \frac{1}{2}at^2$, for a.

19. $s = \frac{1}{2}a(2t - 1)$, for t.

20. Any sort of a bar resting on a fixed point or edge is called a **lever**; the point or edge is called the **fulcrum.**

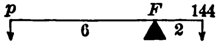
A lever will just balance when the numerical product of the power (p) and its distance (d) from the fulcrum (F) is equal to the numerical product of the weight (W) and its distance (D) from the fulcrum; that is, when

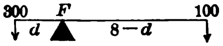


$$pd = WD.$$

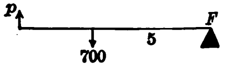
Solve for W and find what weight a power of 150 (pounds) will support by means of the lever shown, if $d = 7$ (feet) and $D = 3$ (feet).

Find for what values of p, d, W, or D the following levers will balance, each lever being 8 feet long:

21. 

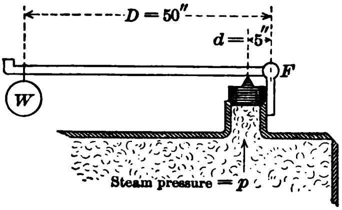
22. 

23. 

24. 

25. Philip, who weighs 114 pounds, and William, who weighs 102 pounds, are balanced on the ends of a 9-foot plank. Neglecting the weight of the plank, how far is Philip from the fulcrum?

26. The figure illustrates the lever of a safety valve, the power being the steam pressure (p) acting on the end of the piston above. The area of the end of the piston is 16 square inches. What weight (W) must be hung on the end of the lever so that when the steam pressure rises to 100 pounds per square inch the piston will rise and allow steam to escape?



Solve:

27. $\dfrac{E}{E'} = \dfrac{R}{R'}$, for R.

28. $I = \dfrac{E}{R + r}$, for r.

29. $a = \dfrac{v_1 - v_0}{t}$, for v_1.

30. $V = V_0\left(1 + \dfrac{t}{273}\right)$, for t.

31. Solve $\dfrac{E}{e} = \dfrac{R + r}{r}$, for R; for r.

32. Solve $\dfrac{1}{f} = \dfrac{1}{f_1} + \dfrac{1}{f_2}$, for f_1; for f_2.

33. Solve $\frac{1}{8}Wl = \dfrac{SI}{c}$, for W; for S; for $\dfrac{I}{c}$.

34. Solve $C = \dfrac{1}{C_1} + \dfrac{1}{C_2}$, for C_1; for C_2.

35. The number of pounds pressure (P) on A square feet of surface of any body submerged to a depth of h feet in a liquid that weighs w pounds per cubic foot is given by the formula

$$P = wAh.$$

Fresh water weighs about $62\frac{1}{2}$ pounds per cubic foot, and ordinary sea water about 64 pounds per cubic foot.

Find the pressure on 1 square foot of surface at the bottom of a standpipe in which the water is 30 feet high; at the bottom of the ocean at a depth of 3000 feet.

36. Solve $P = wAh$ for h and find the value of h when $P = 5000$, $w = 62\frac{1}{2}$, and $A = 8$.

37. At what depth in fresh water will the pressure on an object having a total area of 4 square feet be 2000 pounds?

38. The bottom of a rectangular cistern is 6 feet square. For what depth of water will the pressure on the bottom be 36,000 pounds?

39. How deep in the ocean can a diver go, without danger, in a suit of armor that can sustain safely a pressure of 140 pounds per square inch (20,160 pounds per square foot)?

40. If the pressure per square foot on the bottom of a tank holding 18 feet of petroleum is 990 pounds, what is the weight of the petroleum per cubic foot?

41. The side of a chest lying in 25 feet of water was 5 square feet in area, and sustained a pressure of 8000 pounds. Was the chest submerged in fresh or in salt water?

42. The pressure on the inner surface of a water pipe is 60 pounds per square inch at the faucet in the basement of a house and 40 pounds per square inch at the faucet in the top story. How much higher is the faucet in the top story than the one in the basement?

SUGGESTION. — The pressure due alone to the height of the upper faucet above the lower one is 60 pounds less 40 pounds, or 20 pounds, per square inch, or 2880 pounds per square foot.

SIMULTANEOUS SIMPLE EQUATIONS

TWO UNKNOWN NUMBERS

238. In the equation $x + y = 12$,

x and y may have an unlimited number of pairs of values,

as $x = 1$ and $y = 11$;

or $x = 2$ and $y = 10$; etc.

For since $y = 12 - x$,

if any value is assigned to x, a corresponding value of y may be obtained.

An equation that is satisfied by an unlimited number of sets of values of its unknown numbers is called an **indeterminate equation.**

239. Principle. — *Any single equation involving two or more unknown numbers is indeterminate.*

240. The equations
$$\left.\begin{aligned} 2x + 2y &= 10 \\ 3x + 3y &= 15 \end{aligned}\right\}$$
(and)

express but one relation between x and y; namely, that their sum is 5. In fact, the equations are *equivalent* to

$$x + y = 5$$

and to each other. Such equations are often called **dependent equations,** for either may be *derived* from the other.

241. The equations
$$\left.\begin{aligned} x + y &= 5 \\ x - y &= 1 \end{aligned}\right\}$$

express two distinct relations between x and y, namely, that

sum is 5 and their difference is 1. The equations cannot be ;ed to the same equation; that is, they are not *equivalent.* uations that express different relations between the un- 'n numbers involved, and so cannot be reduced to the equation, are called **independent equations.**

2. Each of the equations

$$\left.\begin{array}{l} x + y = 5 \\ x - y = 1 \end{array}\right\}$$

;isfied separately by an unlimited number of sets of values and y, but they have only one set of values in common, ely,

$$x = 3 \text{ and } y = 2.$$

wo or more equations that are satisfied by the same set or of values of the unknown numbers form a **system of ltaneous, or consistent, equations.**

3. The equations $\left.\begin{array}{l} x + y = 5 \\ x + y = 7 \end{array}\right\}$

; no set of values of x and y in common.

ich equations are called **inconsistent equations.**

14. The student is familiar with the methods of solving ple equations involving *one* unknown number. The gen- method of solving a system of two independent simul- ous simple equations in *two* unknown numbers,

$$\left.\begin{array}{l} x + y = 5 \\ x - y = 3 \end{array}\right\}$$

) combine the equations, using axioms 1–4 in such a way o obtain an equation involving x alone, and another in- ing y alone, which may be solved separately by previous hods.

he process of deriving from a system of simultaneous equa- s another system involving fewer unknown numbers is ed *elimination.*

Elimination by Addition or Subtraction

245. In solving simultaneous equations we may apply axioms 1–4, subject to the restrictions mentioned in § 230 in regard to the introduction or removal of roots in multiplying or dividing both members by expressions involving unknown numbers.

$$\begin{array}{rl} & 5x+2y=24 \\ & \underline{5x-2y=16} \\ \text{Adding,} & 10x \qquad =40 \end{array} \qquad \begin{array}{rl} & 5x+2y=24 \\ & \underline{5x-2y=16} \\ \text{Subtracting,} & \qquad 4y=\ 8 \end{array}$$

In the two given equations the coefficients of y are numerically equal and opposite in sign. Therefore, if the equations are added (Ax. 1), the resulting equation will not involve y. This method of eliminating y illustrates **elimination by addition.**

If one equation is subtracted from the other (Ax. 2), the resulting equation will not involve x. The second process illustrates **elimination by subtraction.**

EXERCISES

246. 1. Solve the equations $2x+3y=7$ and $3x+4y=10$.

SOLUTION

$$\begin{cases} 2x+\ \ 3y=7, & (1) \\ 3x+\ \ 4y=10. & (2) \end{cases}$$

$(1)\times 4$,	$8x+12y=28.$	(3)
$(2)\times 3$,	$9x+12y=30.$	(4)
$(4)-(3)$,	$x=2.$	(5)
Substituting (5) in (1),	$4+\ 3y=7.$	
	$\therefore\ y=1.$	

To *verify*, substitute 2 for x and 1 for y in the given equations.

RULE. — *If necessary, multiply or divide the equations by such numbers as will make the coefficients of the quantity to be eliminated numerically equal.*

Eliminate by addition if the resulting coefficients have unlike signs, or by subtraction if they have like signs.

Solve by addition or subtraction, and verify results:

2. $\begin{cases} 7x-5y=52, \\ 2x+5y=47. \end{cases}$

3. $\begin{cases} 3x+2y=23, \\ x+y=8. \end{cases}$

4. $\begin{cases} 3x-4y=7, \\ x+10y=25. \end{cases}$

5. $\begin{cases} 2x-10y=15, \\ 2x-4y=18. \end{cases}$

6. $\begin{cases} 3u-v=4, \\ u+3v=-2. \end{cases}$

7. $\begin{cases} 4x-y=19, \\ x+3y=21. \end{cases}$

8. $\begin{cases} l+2r=5, \\ 2l+r=1. \end{cases}$

9. $\begin{cases} 2a+3b=17, \\ 3a+2b=18. \end{cases}$

10. $\begin{cases} 7s-9v=6, \\ s+2v=14. \end{cases}$

11. $\begin{cases} 13t-u=20, \\ 4t+2u=20. \end{cases}$

12. $\begin{cases} 3d+4y=25, \\ 4d+3y=31. \end{cases}$

13. $\begin{cases} 5p+6q=32, \\ 7p-3q=22. \end{cases}$

14. $\begin{cases} 3a+6z=39, \\ 9a-4z=51. \end{cases}$

15. $\begin{cases} 8x-3y=44, \\ 7x-5y=29. \end{cases}$

16. $\begin{cases} 6x-5y=33, \\ 4x+4y=44. \end{cases}$

17. $\begin{cases} 3m+11n=67, \\ 5m-3n=5. \end{cases}$

Elimination by Comparison

247. If $$x=8-y, \quad (1)$$
and also $$x=2+y, \quad (2)$$
by axiom 5, the two expressions for x must be equal.

$$\therefore 8-y=2+y.$$

By *comparing* the values of x in the given equations, (1) and (2), we have eliminated x and obtained an equation involving y alone.

This method is called **elimination by comparison.**

EXERCISES

248. 1. Solve the equations $2x-3y=10$ and $5x+$

SOLUTION

$$\begin{cases} 2x-3y=10, \\ 5x+2y=6. \end{cases}$$

From (1), $$x=\frac{10+3y}{2}.$$

From (2), $$x=\frac{6-2y}{5}.$$

Comparing the values of x in (3) and (4),

$$\frac{10+3y}{2}=\frac{6-2y}{5}.$$

Solving, $$y=-2.$$

Substituting -2 for y in either (3) or (4),

$$x=2.$$

To *verify*, substitute 2 for x and -2 for y in the given equation

RULE. — *Find an expression for the value of the same u number in each equation, equate the two expressions, and s equation thus formed.*

Solve by comparison, and verify results:

2. $\begin{cases} 3x-2y=10, \\ x+y=70. \end{cases}$

3. $\begin{cases} 5x+y=22, \\ x+5y=14. \end{cases}$

4. $\begin{cases} 2x+3y=24, \\ 5x-3y=18. \end{cases}$

5. $\begin{cases} 3x+5y=14, \\ 2x-3y=3. \end{cases}$

6. $\begin{cases} 3v+2y=36, \\ 5v-9y=23. \end{cases}$

7. $\begin{cases} 2s+7t=8, \\ 3s+9t=9. \end{cases}$

8. $\begin{cases} 4u+6v=19, \\ 3u-2v=\frac{9}{2}. \end{cases}$

9. $\begin{cases} 4v+3w=34, \\ 11v+5w=8 \end{cases}$

10. $\begin{cases} 4x-13y=5, \\ 3x+11y=- \end{cases}$

11. $\begin{cases} 18x-3y=4 \\ 1-4x+3y= \end{cases}$

Elimination by Substitution

249. Given $$3x + 2y = 27, \qquad (1)$$

and $$x - y = 4. \qquad (2)$$

On solving (2) for x, its value is found to be $x = 4 + y$.

If $4 + y$ is *substituted* for x in (1), $3x$ will become $3(4 + y)$, and the resulting equation

$$3(4 + y) + 2y = 27 \qquad (3)$$

will involve y only, x having been eliminated.

Solving (3), $$y = 3.$$

Substituting 3 for y in (2), $x = 7$.

This method is called **elimination by substitution.**

Rule. — *Find an expression for the value of either of the unknown numbers in one of the equations.*

Substitute this value for that unknown number in the other equation, and solve the resulting equation.

EXERCISES

250. Solve by substitution, and verify results:

1. $\begin{cases} x - y = 4, \\ 4y - x = 14. \end{cases}$

2. $\begin{cases} x + y = 10, \\ 6x - 7y = 34. \end{cases}$

3. $\begin{cases} 3x - 4y = 26, \\ x - 8y = 22. \end{cases}$

4. $\begin{cases} 6y - 10x = 14, \\ y - x = 3. \end{cases}$

5. $\begin{cases} y + 1 = 3x, \\ 5x + 9 = 3y. \end{cases}$

6. $\begin{cases} 17 = 3x + z, \\ 7 = 3z - 2x. \end{cases}$

7. $\begin{cases} 4y = 10 - x, \\ y - x = 5. \end{cases}$

8. $\begin{cases} 7z - 3x = 18, \\ 2z - 5x = 1. \end{cases}$

9. $\begin{cases} 3 - 15y = -x, \\ 3 + 15y = 4x. \end{cases}$

10. $\begin{cases} 1 - x = 3y, \\ 3(1 - x) = 40 - y. \end{cases}$

251. Three standard methods of elimination have been given. Though each is applicable under all circumstances, in special cases each has its peculiar advantages. The student should endeavor to select the method best adapted or to invent a method of his own.

EXERCISES

252. Solve by any method, verifying all results:

1. $\begin{cases} x+z=13, \\ x-z=5. \end{cases}$

2. $\begin{cases} 3x+y=10, \\ x+3y=6. \end{cases}$

3. $\begin{cases} 4x+5y=-2, \\ 5x+4y=2. \end{cases}$

4. $\begin{cases} 5x-y=28, \\ 3x+5y=28. \end{cases}$

5. $\begin{cases} x+3=y-3, \\ 2(x+3)=6-y. \end{cases}$

6. $\begin{cases} 5x-y=12, \\ x+3y=12. \end{cases}$

7. $\begin{cases} 4(2-x)=3y, \\ 2(2-x)=2(y-2). \end{cases}$

8. $\begin{cases} (x+1)+(y-2)=7, \\ (x+1)-(y-2)=5. \end{cases}$

Eliminate before or after clearing of fractions, as may be more advantageous:

9. $\begin{cases} x+\dfrac{y}{3}=11, \\ \dfrac{x}{3}+3y=21. \end{cases}$

10. $\begin{cases} \dfrac{3x}{4}+\dfrac{4y}{5}=21, \\ \dfrac{2x}{3}+\dfrac{3y}{5}=17. \end{cases}$

11. $\begin{cases} \dfrac{x}{3}=11-\dfrac{y}{2}, \\ \dfrac{x}{3}+\dfrac{2y}{7}=8. \end{cases}$

12. $\begin{cases} \dfrac{x}{2}-\dfrac{2y}{3}=-2, \\ \dfrac{5x}{2}+\dfrac{y}{3}=12. \end{cases}$

13. $\begin{cases} \dfrac{x+y}{2}-\dfrac{x-y}{3}=8, \\ \dfrac{x+y}{3}+\dfrac{x-y}{4}=11. \end{cases}$

14. $\begin{cases} \dfrac{x}{2}-\dfrac{y}{3}-1=0, \\ \dfrac{2x-1}{2}-\dfrac{3y-1}{3}=\dfrac{5}{6}. \end{cases}$

15. $\begin{cases} \dfrac{x}{3}=\dfrac{y}{2}, \\ \dfrac{x}{3}-\dfrac{y}{3}=1. \end{cases}$

16. $\begin{cases} \dfrac{3x}{4}+\dfrac{2y}{3}=20, \\ \dfrac{x}{2}+\dfrac{3y}{4}=17. \end{cases}$

17. $\begin{cases} \dfrac{x-1}{4}+y=3, \\ \dfrac{x-1}{4}+4y=9. \end{cases}$

18. $\begin{cases} \dfrac{x}{8}+4y=15, \\ \dfrac{x}{6}+\dfrac{2y}{3}=6. \end{cases}$

19. $\begin{cases} \dfrac{1}{x-1}-\dfrac{3}{x+y}=0, \\ \dfrac{3}{x-y}+3=0. \end{cases}$

20. $\begin{cases} \dfrac{x}{2}-12=\dfrac{y+32}{4}, \\ \dfrac{y}{8}+\dfrac{3x-2y}{5}=25. \end{cases}$

21. $\begin{cases} \dfrac{7+x}{5}-\dfrac{2x-y}{4}=3y-5, \\ \dfrac{5y-7}{2}+\dfrac{4x-3}{6}=18-5x. \end{cases}$

22. $\begin{cases} \dfrac{.2y+.5}{1.5}=\dfrac{.49x-.7}{4.2}, \\ \dfrac{.5x-.2}{1.6}=\dfrac{41}{16}-\dfrac{1.5y-11}{8}. \end{cases}$

23. $\begin{cases} \dfrac{x+y}{5}+\dfrac{x-y}{5}=\dfrac{2x-y}{8}+\dfrac{4y-x}{5}+1, \\ x=2y. \end{cases}$

24. $\begin{cases} x+\frac{1}{2}(3x-y-1)=\frac{1}{4}+\frac{3}{4}(y-1), \\ \frac{1}{5}(4x+3y)=\frac{1}{10}(7y+24). \end{cases}$

25. $\begin{cases} \dfrac{6x+9}{4}+\dfrac{3x+5y}{4x-6}=3\frac{1}{4}+\dfrac{3x+4}{2}, \\ \dfrac{8y+7}{10}+\dfrac{6x-3y}{2(y-4)}=4+\dfrac{4y-9}{5}. \end{cases}$

26. $\begin{cases} \dfrac{3x-5y}{3}-\dfrac{2x-8y-9}{12}=\dfrac{31}{12}, \\ \left(\dfrac{x}{7}+\dfrac{y}{4}+1\frac{1}{3}\right)-\left(4x-\dfrac{y}{8}-25\right)=\dfrac{5}{6}. \end{cases}$

27. $\begin{cases} \frac{1}{2} R - \frac{1}{3}(r+1) = 1\frac{1}{2}, \\ \frac{1}{3}(R-1) - \frac{1}{2} r = 4\frac{1}{2}. \end{cases}$ **28.** $\begin{cases} 2.4\,d - .04\,u = .62, \\ .7\,u + .15\,d = 1.795. \end{cases}$

29. $\begin{cases} (u+.3)(v+.5) = (u-.3)(v+2), \\ (2\,u+.1)(3\,v+.5) = 6\,v(u+.3). \end{cases}$

30. $\begin{cases} x - 20 - \dfrac{2\,y - x}{23 - x} = \dfrac{2\,x - 59}{2}, \\ y - \dfrac{3 - y}{x - 18} - 30 = \dfrac{3\,y - 73}{3}. \end{cases}$

31. $\begin{cases} \dfrac{x}{2} + \dfrac{16 - x}{2} = 30 + \dfrac{5\,y + 2\,x}{40 - x}, \\ \dfrac{4(x-6)}{y+8} + \dfrac{83 - 8\,y}{8} = 10 - y. \end{cases}$

Equations of the form $\frac{a}{x} + \frac{b}{y} = c$, though not simple equations, may be solved as simple equations for some of their roots by regarding $\frac{1}{x}$ and $\frac{1}{y}$ as the unknown numbers.

32. Solve the equations $\begin{cases} \dfrac{4}{x} - \dfrac{3}{y} = \dfrac{14}{5}, & (1) \\ \dfrac{4}{x} + \dfrac{10}{y} = \dfrac{50}{3}. & (2) \end{cases}$

SOLUTION. (2) — (1), $$\frac{13}{y} = \frac{208}{15}.$$

$$\therefore \frac{1}{y} = \frac{16}{15}. \qquad (3)$$

Substituting (3) in (1), $$\frac{4}{x} - \frac{48}{15} = \frac{14}{5}.$$

$$\therefore \frac{1}{x} = \frac{3}{2}. \qquad (4)$$

From (4) and (3), $$x = \frac{2}{3} \text{ and } y = \frac{15}{16}.$$

and verify results:

$$\ldots - \frac{3}{y} = -2, \quad \frac{25}{x} + \frac{1}{y} = 6.$$

37. $$\begin{cases} \dfrac{5}{x} + \dfrac{6}{y} = 64, \\ \dfrac{6}{x} + \dfrac{5}{y} = 73\tfrac{1}{2}. \end{cases}$$

$$\ldots - \frac{3}{y} = 5, \quad \ldots - \frac{2}{y} = 7.$$

38. $$\begin{cases} \dfrac{3}{2x} - \dfrac{1}{y} = -3, \\ \dfrac{5}{2x} + \dfrac{3}{y} = 23. \end{cases}$$

$$\ldots + \frac{3}{y} = \frac{9}{8}, \quad \ldots + \frac{4}{y} = \frac{11}{12}.$$

39. $$\begin{cases} \dfrac{10}{x} + \dfrac{5}{y} = 20, \\ \dfrac{5}{x} + \dfrac{10}{y} = 57\tfrac{1}{2}. \end{cases}$$

$$\ldots + \frac{8}{y} = 30, \quad \ldots + \frac{8}{x} = 30.$$

40. $$\begin{cases} \dfrac{7}{8x} - \dfrac{2}{3y} = 10, \\ \dfrac{5}{6x} - \dfrac{2}{11y} = 17. \end{cases}$$

he following as if $\dfrac{1}{x-1}, \dfrac{1}{y+1}$, etc., were the unknown and then find the values of x and y:

$$\frac{1}{\ldots-1} + \frac{1}{y+1} = 5, \quad \frac{2}{\ldots-1} + \frac{3}{y+1} = 12.$$

43. $$\begin{cases} \dfrac{1}{y} = \dfrac{3}{2-x}, \\ \dfrac{5}{y} = \dfrac{6}{2-x} + 9. \end{cases}$$

$$\frac{5}{\ldots-1} - \frac{3}{y-1} = 14, \quad \frac{2}{\ldots-1} - \frac{1}{y-1} = 6.$$

44. $$\begin{cases} \dfrac{4}{x} = \dfrac{1}{y+3}, \\ \dfrac{7}{x} = \dfrac{3}{y+3} - 10. \end{cases}$$

Literal Simultaneous Equations

253. 1. Solve the equations $\begin{cases} ax + by = m, \\ cx + dy = n. \end{cases}$

Solution

$$ax + by = m \qquad (1)$$

$$cx + dy = n \qquad (2)$$

(1) × d, $$adx + bdy = dm \qquad (3)$$

(2) × b, $$bcx + bdy = bn \qquad (4)$$

(3) − (4), $$(ad - bc)x = dm - bn$$

$$\therefore x = \frac{dm - bn}{ad - bc}. \qquad (5)$$

(1) × c, $$acx + bcy = cm \qquad (6)$$

(2) × a, $$acx + ady = an \qquad (7)$$

(7) − (6), $$(ad - bc)y = an - cm$$

$$\therefore y = \frac{an - cm}{ad - bc}. \qquad (8)$$

In solving literal simultaneous equations, elimination is usually performed by addition or subtraction.

Solve for x and y, and test results by assigning suitable values to the other letters:

2. $\begin{cases} ax + by = m, \\ bx - ay = c. \end{cases}$

3. $\begin{cases} ax - by = m, \\ cx - dy = r. \end{cases}$

4. $\begin{cases} ax = by, \\ x + y = ab. \end{cases}$

5. $\begin{cases} m(x + y) = a, \\ n(x - y) = 2a. \end{cases}$

6. $\begin{cases} a(x - y) = 5, \\ bx - cy = n. \end{cases}$

7. $\begin{cases} a(a - x) = b(y - b), \\ ax = by. \end{cases}$

8. $\begin{cases} x + y = b - a, \\ bx - ay + 2ab = 0. \end{cases}$

9. $\begin{cases} x - y = a - b, \\ ax + by = a^2 - b^2. \end{cases}$

10. $\begin{cases} \dfrac{x}{a}+\dfrac{y}{b}-2=0, \\ bx-ay=0. \end{cases}$

11. $\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}=\dfrac{1}{a}, \\ \dfrac{1}{x}-\dfrac{1}{y}=\dfrac{1}{b}. \end{cases}$

12. $\begin{cases} \dfrac{a}{x}-\dfrac{b}{y}=-1, \\ \dfrac{b}{x}-\dfrac{a}{y}=-1. \end{cases}$

13. $\begin{cases} \dfrac{x}{a}+\dfrac{y}{b}=2ab, \\ \dfrac{x}{ab}+\dfrac{y}{ab}=a+b. \end{cases}$

14. $\begin{cases} \dfrac{x}{a}+\dfrac{y}{b}=1, \\ \dfrac{x}{b}-\dfrac{y}{a}=\dfrac{1}{2}. \end{cases}$

15. $\begin{cases} \dfrac{x+1}{y+1}=\dfrac{a+b+1}{a-b+1}, \\ x-y=2b. \end{cases}$

16. $\begin{cases} \dfrac{1}{x-a}=\dfrac{1}{a-y}, \\ \dfrac{x+y}{x-y}=a. \end{cases}$

17. $\begin{cases} \dfrac{x}{a}+\dfrac{y}{b}=c, \\ \dfrac{x}{b}+\dfrac{y}{c}=d. \end{cases}$

18. $\begin{cases} \dfrac{a}{x}+\dfrac{b}{y}=c, \\ \dfrac{m}{x}+\dfrac{n}{y}=e. \end{cases}$

19. $\begin{cases} \dfrac{1}{ax}+\dfrac{1}{by}=c, \\ \dfrac{1}{bx}-\dfrac{1}{ay}=d. \end{cases}$

20. Given $\begin{cases} F=Ma, \\ s=\frac{1}{2}at^2. \end{cases}$

Find the values of F and a when $M=15$, $s=72$, and $t=6$.

21. Given $\begin{cases} l=a+(n-1)d, \\ s=\dfrac{n}{2}(a+l). \end{cases}$

Find the values of a and l when $n=50$, $d=2$, and $s=2500$; the values of d and a when $l=50$, $n=25$, and $s=650$.

22. Given $\begin{cases} l=ar^{n-1}, \\ s=\dfrac{rl-a}{r-1}. \end{cases}$

Find the values of a and l when $r=2$, $n=11$, and $s=2047$.

Problems

254. Find two numbers related to each other as follows:

1. Sum $= 14$; difference $= 8$.

2. Sum of 2 times the first and 3 times the second $= 34$; sum of 2 times the first and 5 times the second $= 50$.

3. Sum $= 18$; sum of the first and 2 times the second $= 20$.

4. A grocer sold 2 boxes of raspberries and 3 of cherries to one customer for 54¢, and 3 boxes of raspberries and 2 of cherries to another for 56¢. Find the price of each per box.

5. A druggist wishes to put 500 grains of quinine into 3-grain and 2-grain capsules. He fills 220 capsules. How many capsules of each size does he fill?

6. On the Fourth of July, 850 glasses of soda water were sold at a fountain, some at 5¢ each, the others at 10¢ each. The receipts were $55. How many were sold at each price?

7. A fruit dealer bought 36 pineapples for $2.50. He sold some at 12¢ each and the rest at 10¢ each, thereby gaining $1.50. How many did he sell at each price?

8. The receipts from 300 tickets for a musical recital were $125. Adults were charged 50¢ each and children 25¢ each. How many tickets of each kind were sold?

9. An errand boy went to the bank to deposit some bills for his employer. Some of them were 1-dollar bills, and the rest 2-dollar bills. The number of bills was 38 and their value was $50. Find the number of each.

10. If 2 full-grown rubber trees in Brazil yield 4 pounds more of rubber in a year than 8 trees in Ceylon, and 3 Brazil trees yield 10 pounds more than 10 Ceylon trees, what is the average yield per tree in each country?

11. A man noticed that a 15-word message by telegraph cost him 40¢ and a 22-word message 54¢, between the same two cities. Find the charge for the first ten words and the charge for each additional word.

12. At a factory where 1000 men and women were employed, the average daily wage was \$2.50 for a man and \$1.50 for a woman. If labor cost \$2340 per day, how many men were employed? how many women?

13. It required 60 inches of tape to bind the four edges of a card on which a photograph was mounted. The length of the card was 6 inches greater than the width. How many inches long was the card? how many inches wide?

14. A lieutenant of the U. S. navy received \$150 per month while on sea duty and \$127.50 per month while on shore duty. His salary for a year amounted to \$1620. How many months was he on sea duty? on shore duty?

15. The great columns of Bedford stone in the Indianapolis post office building weigh 94 tons each, including the shafts and the capitals resting on them. Each shaft weighs 74 tons more than its capital. Find the weight of a shaft; of a capital.

16. The receipts from a football game were \$700. Admission tickets to the grounds were sold for 50¢, and to the grand stand, for 25¢ in addition. If twice as many persons had purchased tickets for the grand stand, the receipts would have been \$800. How many tickets of each kind were sold?

17. The duty paid on an importation of 40,000 shingles and 160,000 laths was \$52, and that on 80,000 shingles and 70,000 laths was \$41.50. Find the rate of duty per thousand on each.

18. If 1 is added to the numerator of a certain fraction, its value becomes $\frac{3}{4}$; if 2 is added to the denominator, its value becomes $\frac{1}{2}$. What is the fraction?

SUGGESTION. — Let $\frac{x}{y}$ = the fraction.

19. If each term of a certain fraction is increased by 1, the value of the fraction is decreased by $\frac{1}{20}$; but if each term is decreased by 1, the value of the fraction is increased by $\frac{1}{12}$. What is the fraction?

20. A certain number expressed by two digits is equal to times the sum of its digits; if 27 is subtracted from the nu ber, the difference will be expressed by reversing the order the digits. What is the number?

SUGGESTION. — The sum of x tens and y units is $(10x+y)$ units; y tens and x units, $(10y+x)$ units.

21. The sum of the two digits of a certain number is 1 and the number is 3 greater than 6 times the sum of its digit What is the number?

22. When a certain number expressed by two digits divided by the sum of its digits, the quotient is 8; and wh the first digit is diminished by 3 times the second, the mainder is 1. What is the number?

23. If a rectangular floor were 2 feet wider and 5 fe longer, its area would be 140 square feet greater; if it we 7 feet wider and 10 feet longer, its area would be 390 squa feet greater. What are its dimensions?

24. A crew can row 8 miles downstream and back, or 1 miles downstream and halfway back in $1\frac{1}{2}$ hours. What is thei rate of rowing in still water and the velocity of the stream?

25. A man rows 12 miles downstream and back in 11 hours. The current is such that he can row 8 miles downstream in the same time as 3 miles upstream. What is his rate of rowing in still water, and what is the velocity of the stream?

26. A train of 25 cars loaded with iron ore was run out on a dock and the ore emptied into pockets beneath the tracks. The ore filled 7 pockets and $\frac{1}{2}$ of another. To fill this last pocket, then, required 16 tons less than 2 extra car loads. What was the capacity of a car? of a pocket?

27. If 100 pounds of soft coal in burning can evaporate 50 pounds more water than 6 gallons of oil, and if 60 pounds of coal can evaporate 10 pounds less water than 4 gallons of oil, how many pounds of water can 1 pound of coal evaporate? 1 gallon of oil?

28. A Florida farmer shipped 24,300 pineapples packed in 450 crates of one size and 375 crates of another. Later he shipped 36,000 pineapples in 675 crates of the first size and 550 crates of the second. Find the capacity of a crate of each size.

29. The weight of a quantity of naphtha and petroleum was 12,400 pounds. Each gallon of naphtha weighed $5\frac{3}{4}$ pounds and cost $6\frac{7}{8}$¢; each gallon of petroleum weighed $6\frac{1}{2}$ pounds and cost $7\frac{1}{2}$¢. If the sum paid for the total quantity was $145, how many gallons were there of each product?

30. A German dredge on trial removed in $1\frac{4}{5}$ hours a quantity of mud that at the contract rate would have required $2\frac{1}{2}$ hours, removing 1400 cubic meters more per hour than was required by the contract. Find the contract rate per hour, and the actual rate.

31. A and B together can do a piece of work in 12 days. After A has worked alone for 5 days, B finishes the work in 26 days. In what time can each alone do the work?

32. A quantity of wheat could be thrashed by two machines in 6 days, but the larger machine worked alone for 8 days and was then replaced by the smaller, which finished in 3 days. How long would it have taken the larger machine to thrash all of the wheat? the smaller machine?

33. The plates of a ship can be riveted in 30 days by 10 gangs of riveters, 4 using hand hammers and 6 using pneumatic hammers; or in 32 days by 10 gangs, 5 of each kind. How long would it take 12 gangs all using pneumatic hammers?

34. A and B can do a piece of work in a days, or if A works m days alone, B can finish the work by working n days. In how many days can each do the work?

35. A can build a wall in c days, and B can build it in d days. How many days must each work so that, after A has done a part of the work, B can take his place and finish the wall in a days from the time A began?

36. At simple interest a sum of money amounted to \$2472 in 9 months and to \$2528 in 16 months. Find the amount of money at interest and the rate.

37. A man invested \$4000, a part at 5% and the rest at 4%. If the annual income from both investments was \$175, what was the amount of each investment?

38. A man invested a dollars, a part at r% and the rest at s% yearly. If the annual income from both investments was b dollars, what was the amount of each investment?

39. A sum of money at simple interest amounted to b dollars in t years, and to a dollars in s years. What was the principal, and what was the rate of interest?

40. A certain number of people charter an excursion boat, agreeing to share the expense equally. If each pays a cents, there will be b cents lacking from the necessary amount; and if each pays c cents, d cents too much will be collected. How many persons are there, and how much should each pay?

41. A mine is emptied of water by two pumps which together discharge m gallons per hour. Both pumps can do the work in b hours, or the larger can do it in a hours. How many gallons per hour does each pump discharge? What is the discharge of each per hour when $a = 5$, $b = 4$, and $m = 1250$?

42. Two trains are scheduled to leave A and B, m miles apart, at the same time, and to meet in b hours. If the train that leaves B is a hours late and runs at its customary rate, it will meet the first train in c hours. What is the rate of each train?

What is the rate of each, if $m = 800$, $c = 9$, $a = 1\frac{4}{5}$, and $b = 10$?

43. A man ordered a certain amount of cement and received it in c barrels and d bags; a barrels and b bags made $\frac{m}{n}$ of the total weight. How many barrels or how many bags alone would have been needed? Find the number of each, if $c = 16$, $d = 15$, $a = 6$, $b = 15$, $m = 1$, and $n = 2$.

'HREE OR MORE UNKNOWN NUMBERS

ıe student has been solving systems of *two* independ- ;aneous equations involving *two* unknown numbers.

·,

›LE. — *Every system of independent simultaneous simple 'nvolving the same number of unknown numbers as there ons can be solved, and is satisfied by one and only one es of its unknown numbers.*

EXERCISES

$$\text{Solve}\begin{cases} x+2y+3z=14, & (1)\\ 2x+y+2z=10, & (2)\\ 3x+4y-3z=2. & (3)\end{cases}$$

. — Eliminating z by combining (1) and (3),

$$4x+6y=16. \qquad (4)$$

ng z by combining (2) and (3),

$$\begin{array}{r} 6x+\ \ 3y+6z=30\\ 6x+\ \ 8y-6z=\ \ 4\\ \hline 12x+11y\qquad\ \ =34\end{array} \qquad (5)$$

ng x by combining (5) and (4),

$$12x+18y=48 \qquad (6)$$

$$7y=14;\ \therefore y=2.$$

.ng the value of y in (4), $4x+12=16$; $\therefore x=1$.

.ng the values of x and y in (1),

$$1+4+3z=14;\ \therefore z=3.$$

.TION. — Substituting $x=1$, $y=2$, and $z=3$ in the given

(1) becomes $1+4+9=14$, or $14=14$;
(2) becomes $2+2+6=10$, or $10=10$;
(3) becomes $3+8-9=\ 2$, or $2=\ 2$;

iven equations are satisfied for $x=1$, $y=2$, and $z=3$.

Solve, and test all results:

2. $$\begin{cases} x+3y-z=10, \\ 2x+5y+4z=57, \\ 3x-y+2z=15. \end{cases}$$

3. $$\begin{cases} x+y+z=53, \\ x+2y+3z=105, \\ x+3y+4z=134. \end{cases}$$

4. $$\begin{cases} x-y+z=30, \\ 3y-x-z=12, \\ 7z-y+2x=141. \end{cases}$$

5. $$\begin{cases} 8x-5y+2z=53, \\ x+y-z=9, \\ 13x-9y+3z=71. \end{cases}$$

6. $$\begin{cases} x+3y+4z=83, \\ x+y+z=29, \\ 6x+8y+3z=156. \end{cases}$$

7. $$\begin{cases} 2x+3y+4z=29, \\ 3x+2y+5z=32, \\ 4x+3y+2z=25. \end{cases}$$

8. $$\begin{cases} 3x-2y+z=2, \\ 2x+5y+2z=27, \\ x+3y+3z=25. \end{cases}$$

9. $$\begin{cases} 2x-3y+4z-v=4, \\ 4x+2y-z+2v=13, \\ x-y+2z+3v=17, \\ 3x+2y-z+4v=20. \end{cases}$$

10. $$\begin{cases} 4x-5y+3z=14, \\ x+7y-z=13, \\ 2x+5y+5z=36. \end{cases}$$

11. $$\begin{cases} 2x+y-3z+4w=44, \\ 3x-2y+z-w=-1, \\ 4x-y+2z+w=55, \\ 5x-3y+4z-w=39. \end{cases}$$

12. $$\begin{cases} 7x-1=3y, \\ 11z-1=7v, \\ 4z-1=7y, \\ 19x-1=3v. \end{cases}$$

13. $$\begin{cases} x+\frac{1}{2}y+\frac{1}{3}z=32, \\ \frac{1}{3}x+\frac{1}{4}y+\frac{1}{5}z=15, \\ \frac{1}{4}x+\frac{1}{5}y+\frac{1}{6}z=12. \end{cases}$$

14. $$\begin{cases} \frac{1}{6}x-\frac{1}{5}y+\frac{1}{4}z=3, \\ \frac{1}{5}x-\frac{1}{4}y+\frac{1}{5}z=1, \\ \frac{1}{4}x-\frac{1}{3}y+\frac{1}{2}z=5. \end{cases}$$

15. $$\begin{cases} \dfrac{x+y}{3}+3z=29, \\ \dfrac{2x-y}{2}+2z=22, \\ 3x-y=3(z-1). \end{cases}$$

16. $$\begin{cases} 3x+y-z+2v=0, \\ 3y-2x+z-4v=2, \\ x-y+2z-3v=6, \\ 4x+2y-3z+v=12 \end{cases}$$

e the equations $\begin{cases} u+v+x-y=2, \\ u+v-x+y=4, \\ u-v+x+y=6, \\ v-u+x+y=8. \end{cases}$

— Adding the equations, $2u+2v+2x+2y=20$.

y 2, $u+v+x+y=10$.

g each of the given equations from this equation,

$$2y=8,\quad 2x=6,\quad 2v=4,\quad 2u=2;$$

$$y=4,\quad x=3,\quad v=2,\quad u=1.$$

d test all results:

$y=9,$
$z=7,$
$x=5.$

22. $\begin{cases} x+3y+z=14, \\ x+y+3z=16, \\ 3x+y+z=20. \end{cases}$

$x+y=15,$
$y+z=18,$
$z+v=17,$
$v+x=16.$

23. $\begin{cases} y+z+v-x=22, \\ z+v+x-y=18, \\ v+x+y-z=14, \\ x+y+z-v=10. \end{cases}$

$\frac{1}{y}=6,$
$\frac{1}{z}=10,$
$\frac{1}{x}=8.$

24. $\begin{cases} \frac{1}{x}+\frac{1}{y}-1=0, \\ \frac{1}{y}+\frac{1}{z}+3=0, \\ \frac{1}{z}+\frac{1}{x}-2=0. \end{cases}$

$\frac{}{y}=\frac{1}{5},$
$\frac{}{z}=\frac{1}{6},$
$\frac{}{x}=\frac{1}{7}.$

25. $\begin{cases} \frac{xy}{x+y}=\frac{1}{8}, \\ \frac{yz}{y+z}=\frac{1}{4}, \\ \frac{zx}{z+x}=\frac{1}{2}. \end{cases}$

.—*If* $\frac{xy}{x+y}=\frac{1}{5}$, $\frac{x+y}{xy}=\frac{5}{1}$; whence, $\frac{1}{y}+\frac{1}{x}=5$.

GRAPHIC SOLUTIONS

SIMPLE EQUATIONS

258. When related quantities in a series are to be compared, as for instance the population of a town in successive years, recourse is often had to a method of representing quantities by *lines*. This is called the **graphic** method.

By this method, quantity is photographed in the process of change. The whole range of the variation of a quantity, presented in this vivid pictorial way, is easily comprehended at a glance; it stamps itself on the memory.

259. In Fig. 1 is shown the population of a town throughout its variations during the first 13 years of the town's existence.

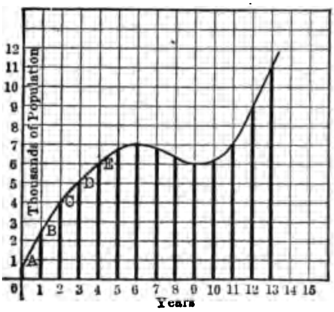


FIG. 1.

The population at the end of 2 years, for example, is represented by the length of the heavy black line drawn upward from 2, and is 4000; the population at the end of 6 years is 7000; at the end of 10 years, 6300 approximately; and so on.

260. Every point of the curved line shown in Fig. 1 exhibits a pair of corresponding values of two related quantities, years and population. For instance, the position of E shows that the population at the end of 4 years was 6000.

Such a line is called a **graph.**

Graphs are useful in numberless ways. The statistician uses them to sent information in a telling way. The broker or merchant uses them compare the rise and fall of prices. The physician uses them to record progress of diseases. The engineer uses them in testing materials and computing. The scientist uses them in his investigations of the laws nature. In short, graphs may be used whenever two related quantities to be compared throughout a series of values.

261. The graph in Fig. 2 represents the rate in gallons per y per person at which water was used in New York City ıring a certain day of 24 hours.

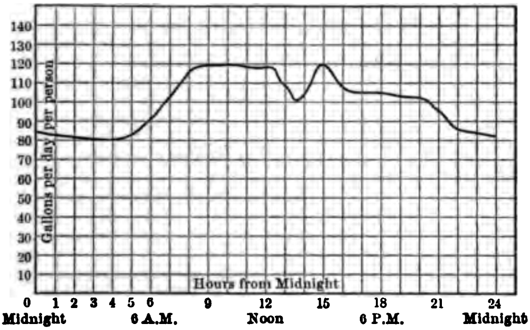


FIG. 2.

Thus, if each horizontal space represents 1 hour (from midnight) and each vertical space 10 gallons, at midnight water was being used at the rate of about 84 gallons per day per person; at 6 A.M., about 91 gallons; at 1 P.M., the 13th hour, about 108 gallons; etc.

1. What was the approximate consumption of water at 2 A.M.? at noon? at 1 : 30 P.M.? at 2 : 30 P.M.? at 6 P.M.?

2. What was the maximum rate during the day? the minimum rate? at what time did each occur?

3. During what hours was the rate most uniform? What was the rate at the middle of each hour?

4. What was the average increase per hour between 6 A.M. and 8 A.M.? the average decrease between 4 P.M. and 8 P.M.?

262. Fig. 3 gives a part of the graph that shows tion between numbers and their : squares, horizontal distances re] the numbers 1, 2, 3, etc., and ve tances, their squares. From this may read the squares or the sqı of various numbers.

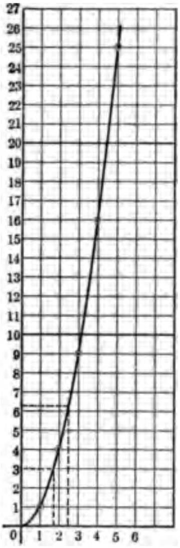


FIG. 3.

Thus, the *square* of 3 is represen vertical line extending from 3 to t It is 9. Conversely, the *square ro* represented by the horizontal line tends from 9 to the graph. It is 3

Similarly, the square of $2\frac{1}{2}$ is re by a vertical line halfway betweeı and extending to the graph. It is square root of 3 is represented by tal line from 3 to the graph. It i: mately 1.7.

Find from the graph the squ of $1\frac{1}{2}$; of $3\frac{1}{4}$; the approximate sq of 11; of 13; of 8; of $2\frac{1}{2}$.

263. Let x and y be two *algebraic* quantities so re: $y = 2x - 3$. It is evident that we may give x a values, and obtain a corresponding series of values that *the number of such pairs of values of x and y is unlimited.* All of these values are represented in the graph of $y = 2x - 3$. Just as in the preceding illustrations, so in the graph of $y = 2x - 3$, Fig. 4, values of x are represented by lines laid off on or parallel to an **x-axis**, X′X, and values of y by lines laid off on or parallel to a **y-axis**, Y′Y, usually drawn perpendicular to the *x-axis.*

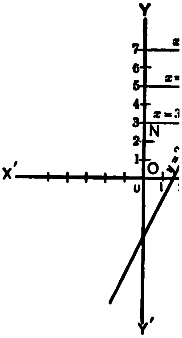


FIG. 4.

For example, the position of P shows that $y=3$ when $x=3$; the position of Q shows that $y=5$ when $x=4$; the position of R shows that $y=7$ when $x=5$; etc.

Evidently every point of the graph gives a pair of corresponding values of x and y.

264. Conversely, to locate any point with reference to two axes for the purpose of representing a pair of corresponding values of x and y, the value of x may be laid off on the x-axis as an *x-distance*, or **abscissa**, and that of y on the y-axis as a *y-distance*, or **ordinate**. If from each of the points on the axes obtained by these measurements, a line parallel to the other axis is drawn, the intersection of these two lines locates the point.

Thus, in Fig. 4, to represent the corresponding values $x=3$, $y=3$, a point P may be located by measuring 3 units from O to M on the x-axis and 3 units from O to N on the y-axis, and then drawing a line from M parallel to OY, and one from N parallel to OX, producing these lines until they intersect.

265. The abscissa and ordinate of a point referred to two perpendicular axes are called the **rectangular coördinates**, or simply the **coördinates**, of the point.

Thus, in Fig. 4, the coördinates of P are $OM\,(=NP)$ and $MP\,(=ON)$.

266. By universal custom *positive* values of x are laid off from O as a zero-point, or **origin**, toward the *right*, and *negative* values toward the *left*. Also *positive* values of y are laid off *upward* and *negative* values *downward*.

The point A in Fig. 5 may be designated as "the point (2, 3)," or by the equation $A=(2, 3)$.

Similarly,

$B=(-2, 4)$, $C=(-3, -1)$, and $D=(1, -2)$.

The abscissa is always written first.

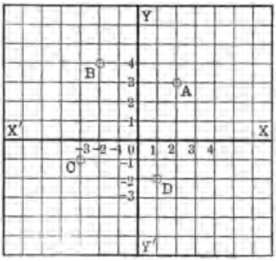


FIG. 5.

267. Plotting points and constructing graphs.

EXERCISES

NOTE. — The use of paper ruled in small squares, called coördinate paper, is advised in plotting graphs.

Draw two axes at right angles to each other and locate:

1. $A = (3, 2)$.
2. $B = (3, -2)$.
3. $C = (4, 3)$.
4. $D = (4, -3)$.
5. $E = (5, 5)$.
6. $F = (-5, 5)$.
7. $G = (-2, 5)$.
8. $H = (-3, -4)$.
9. $L = (0, 4)$.
10. $M = (0, -5)$.
11. $N = (3, 0)$.
12. $P = (-6, 0)$.

13. Where do all points having the abscissa 0 lie? the ordinate 0?

14. What are the coördinates of the origin?

15. Construct the graph of the equation $2y - x = 2$.

SOLUTION

Solving for y, $y = \frac{1}{2}(x + 2)$.

Values are now given to x and computed for y by means of this equation. The numbers substituted for x need not be large. Convenient numbers to be substituted for x in this instance are the even integers from -6 to $+6$.

When $x = -6$, $y = -2$. These values locate the point $A = (-6, -2)$.
When $x = -4$, $y = -1$. These values serve to locate $B = (-4, -1)$.
Other points may be located in the same way.

A record of the work should be kept as follows:

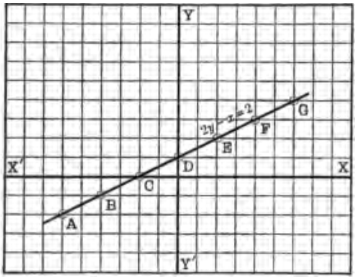


FIG. 6.

$y = \frac{1}{2}(x + 2)$

x	y	POINT
-6	-2	A
-4	-1	B
-2	0	C
0	1	D
2	2	E
4	3	F
6	4	G

A line drawn through A, B, C, D, etc., is the graph of $2y - x = 2$.

the graph of each of the following:

$c-7$.	19. $3x-y=4$.	22. $3x=2y$.
$c+1$.	20. $4x-y=10$.	23. $2x+y=1$.
$c-1$.	21. $x-2y=2$.	24. $2x+3y=6$.

n be proved by the principle of the similarity of t:

.— *The graph of a simple equation is a straight line.* ason simple equations are sometimes called **linear**

a straight line is determined by two points, to ph of a linear equation, *plot two points and draw through them.*

convenient to plot the points where the graph axes. To find where it intersects the x-axis, let l where it intersects the y-axis, let $x=0$.

$\frac{1}{2}(x+2)$, when $y=0$, $x=-2$, locating C, Fig. 6; when cating D.
ght line through C and D.
ion has no absolute term, $x=0$ when $y=0$, and this nly one point. In any case it is desirable, *for the sake of* ot points some distance apart, as A and G, in Fig. 6.

EXERCISES

truct the graph of each of the following:

	8. $2x-3y=6$.	15. $8x-3y=-6$.
	9. $3x+4y=12$.	16. $-2x+y=-3$.
x.	10. $5x-2y=10$.	17. $-3x+4y=8$.
9.	11. $7x-y=14$.	18. $5x+3y=7\frac{1}{2}$.
x.	12. $4-x=2y$.	19. $x-\frac{1}{2}y=3$.
10.	13. $2x+3y=0$.	20. $\frac{1}{2}x+\frac{1}{3}y=2$.
-4.	14. $x-4y-3=0$.	21. $.7x-.3y=4$.

271. Graphic solution of simultaneous linear equations.

I. Let it be required to solve graphically the equations

$$\begin{cases} y = 2 + x, & (1) \\ y = 6 - x. & (2) \end{cases}$$

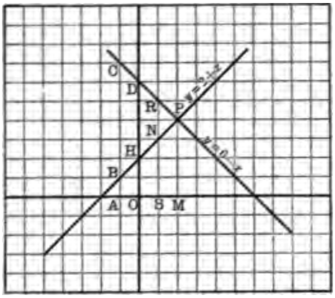


FIG. 7.

As in § 267, construct the graph of each equation, shown in Fig. 7.

1. When $x = -1$, the value of y in (1) is represented by AB, and in (2) by AC.

Therefore, when $x = -1$, the equations are not satisfied by the same values of y.

2. Compare the values of y when $x = 0$; when $x = 1$; 2.

3. For what value of x are the values of y in the two equations equal, or coincident?

4. What values of x and y will satisfy *both* equations?

The required values of x and y, then, are represented graphically by the coördinates of P, *the intersection of the graphs.*

II. Let the given equations be

$$\begin{cases} x + y = 7, \\ 2x + 2y = 14. \end{cases}$$

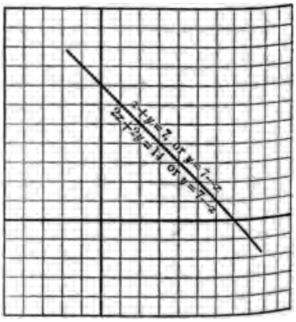


FIG. 8.

5. What happens if we try to eliminate either x or y?

6. Since $y = 7 - x$ in both equations, what will be the relative positions of any two points plotted for the same value of x? the relative positions of the two graphs?

7. The algebraic analysis shows that the equations are **indeterminate.**

The graphic analysis also shows that the equations are *indeterminate, for their graphs* coincide.

III. Let the given equations

$$\begin{cases} y = 6 - x, & (1) \\ y = 4 - x. & (2) \end{cases}$$

8. When $x = -1$, how much eater is the value of y in (1) an in (2), as shown both by e equations and their graphs?

9. Compare the y's for other lues of x.

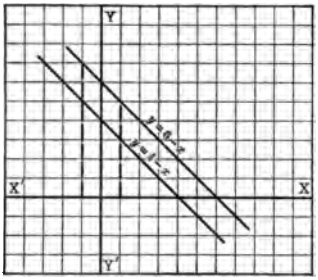


FIG. 9.

10. For every value of x the values of y in the two equa- ms differ by 2, and the graphs are 2 units apart, vertically.

In algebraic language, the equations cannot be simultaneous; at is, they are **inconsistent**.

In graphical language, their graphs *cannot intersect*, being rallel straight lines.

272. PRINCIPLES. — 1. *A single linear equation involving o unknown numbers is indeterminate.*

2. *Two linear equations involving two unknown numbers e determinate, provided the equations are independent and multaneous.*

They are satisfied by one, and only one, pair of common values.

3. *The pair of common values is represented graphically by e coördinates of the intersection of their graphs.*

EXERCISES

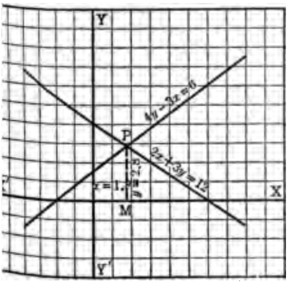


FIG. 10.

273. 1. Solve graphically the equations $\begin{cases} 4y - 3x = 6, \\ 2x + 3y = 12. \end{cases}$

SOLUTION. — On plotting the graphs of both equations, as in § 267, it is found that they intersect at a point P, whose coördinates are 1.8 and 2.8, approximately.

Hence, $x = 1.8$ and $y = 2.8$.

The coördinates of P are estimated to the nearest tenth.

Note. — In solving simultaneous equations by the graphic method the same axes must be used for the graphs of both equations.

Construct the graphs of each of the following systems of equations. Solve, if possible. If there is no solution, tell why.

2. $\begin{cases} x-y=1, \\ x+y=9. \end{cases}$

3. $\begin{cases} x+y=3, \\ x+2y=4. \end{cases}$

4. $\begin{cases} x=4+y, \\ y=3+x. \end{cases}$

5. $\begin{cases} 2x-y=5, \\ 4x+y=16. \end{cases}$

6. $\begin{cases} 3x=y+9, \\ 2y=6x-18. \end{cases}$

7. $\begin{cases} y=4x, \\ x-y=3. \end{cases}$

8. $\begin{cases} x=\frac{1}{2}(y+4), \\ y=2(x-2). \end{cases}$

9. $\begin{cases} x+y=-3, \\ x-2y=-12. \end{cases}$

10. $\begin{cases} x+y=4, \\ y=2-x. \end{cases}$

11. $\begin{cases} x=2(y+1), \\ 21=2(2x+y). \end{cases}$

12. $\begin{cases} x+y=8, \\ 2x-6y=-9. \end{cases}$

13. $\begin{cases} 2x-5y=5, \\ 10y=2x+1. \end{cases}$

14. $\begin{cases} 3y=2x-7, \\ 2x=6+3y. \end{cases}$

15. $\begin{cases} 3(x-4)=2y, \\ 6(y+6)=9x. \end{cases}$

16. $\begin{cases} 10x+y=14, \\ 8x-5y=-2. \end{cases}$

17. $\begin{cases} 2x+3y=8, \\ 3x+2y=8. \end{cases}$

18. $\begin{cases} 4y+3x=5, \\ 4x-3y=3. \end{cases}$

19. $\begin{cases} x+3y=-6, \\ 2x-4y=-12. \end{cases}$

20. $\begin{cases} 4x-10y=0, \\ 2x+y=12. \end{cases}$

21. $\begin{cases} x-2y=2, \\ 2y-6x=3. \end{cases}$

22. $\begin{cases} 3x+4y=10, \\ 6x+8y=20. \end{cases}$

23. $\begin{cases} \frac{3}{2}x+\frac{5}{2}y=3\frac{1}{2}, \\ 10x-2y=14. \end{cases}$

INVOLUTION

274. The process of finding any required power of an expression is called **involution**.

275. By the definition of a power, when n is a positive integer a^n means $a \cdot a \cdot a \cdots$ to n factors.

The following illustrate powers of positive numbers, of negative numbers, of powers, of products, and of quotients, and show that every case of involution is an example of multiplication of *equal* factors.

POWERS OF A POSITIVE NUMBER

$$\begin{array}{r}
2 = 2^1 \\
2 \\
\hline
4 = 2^2 \\
2 \\
\hline
8 = 2^3 \\
2 \\
\hline
16 = 2^4
\end{array}$$

POWERS OF A NEGATIVE NUMBER

$$\begin{array}{r}
-2 = (-2)^1 \\
-2 \\
\hline
4 = (-2)^2 \\
-2 \\
\hline
-8 = (-2)^3 \\
-2 \\
\hline
16 = (-2)^4
\end{array}$$

POWERS OF A POWER

$$\begin{array}{r}
4 = 2^2 \\
4 \\
\hline
16 = (2^2)^2 = 2^4 \\
4 \\
\hline
64 = (2^2)^3 = 2^6 \\
4 \\
\hline
256 = (2^2)^4 = 2^8
\end{array}$$

POWER OF A PRODUCT

$$(2 \cdot 3)^2 = (2 \cdot 3) \times (2 \cdot 3) = 2 \cdot 2 \cdot 3 \cdot 3 = 2^2 \cdot 3^2.$$

POWER OF A QUOTIENT

$$\left(\frac{2}{3}\right)^2 = \frac{2}{3} \cdot \frac{2}{3} = \frac{2^2}{3^2}.$$

The last *two examples* illustrate the distributive law for *involution*

276. Principles. — 1. **Law of Signs for Involution.** — *All powers of a positive number are positive; even powers of a negative number are positive, and odd powers are negative.*

2. **Law of Exponents for Involution.** — *The exponent of a power of a number is equal to the exponent of the number multiplied by the exponent of the power to which the number is to be raised.*

3. **Distributive Law for Involution.** — *Any power of a product is equal to the product of its factors each raised to that power.*

Any power of the quotient of two numbers is equal to the quotient of the numbers each raised to that power.

The above laws may be established for *positive integral exponents* as follows:

Let m and n be positive integers.

1. Principle 1 follows from the law of signs for multiplication.

2. By notation, § 27, $(a^m)^n = a^m \cdot a^m \cdot a^m \cdots$ to n factors

§ 88, $= a^{m+m+m+\cdots \text{ to } n \text{ terms}}$

By notation, $= a^{mn}$.

3. By notation, $(ab)^n = ab \times ab \times ab \cdots$ to n factors

§ 82, $= (aaa \cdots)(bbb \cdots)$ each to n factors

By notation, $= a^n b^n$.

Also $\left(\frac{a}{b}\right)^n = \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} \cdots$ to n factors

§ 207, $= \frac{aaa \cdots \text{ to } n \text{ factors}}{bbb \cdots \text{ to } n \text{ factors}}$

By notation, $= \frac{a^n}{b^n}$.

277. Axiom 6. — *The same powers of equal numbers are equal.*

Thus, if $x = 3$, $x^2 = 3^2$, or 9; also $x^4 = 3^4$, or 81; etc.

278. Involution of monomials.

EXERCISES

1. What is the third power of $4a^3b$?

Solution. $(4a^3b)^3 = 4a^3b \times 4a^3b \times 4a^3b = 64a^9b^3$.

2. What is the fifth power of $-2ab^2$?

SOLUTION

$$(-2ab^2)^5 = -2ab^2 \times -2ab^2 \times -2ab^2 \times -2ab^2 \times -2ab^2 = -32a^5b^{10}.$$

To raise an integral term to any power:

RULE. — *Raise the numerical coefficient to the required power and annex to it each letter with an exponent equal to the product of its exponent by the exponent of the required power.*

Make the power positive or negative according to the law of signs.

Raise to the power indicated:

3. $(ab^2c^3)^2$.	12. $(-4c^2y^5)^3$.	21. $(-1)^{99}$.
4. $(a^3b^2c)^4$.	13. $(-2a^3n^5)^4$.	22. $(-1)^{200}$.
5. $(2a^2c)^3$.	14. $(abcx)^m$.	23. $(-1)^{2n}$.
6. $(7a^2m^5)^2$.	15. $(2e^2x^c)^6$.	24. $(-b)^{2n+1}$.
7. $(-1)^2$.	16. $(3bc)^n$.	25. $(-b^2)^{2n+1}$.
8. $(-ab)^2$.	17. $(2a^2x^3)^n$.	26. $(-a^xb^yc^{n-1}d)^2$.
9. $(-3c)^3$.	18. $(-2l^4m^5d^2)^3$.	27. $(-a^{2n}y^{3p}z^{4r})^5$.
10. $(-10x^2)^3$.	19. $(-a^2x^ny^{n-1})^2$.	28. $(-a^{n-1}b^{n-2}c)^3$.
11. $(-6a^2x^3)^2$.	20. $(-x^3y^3z^{n-3})^3$.	29. $[-2(a-b)^2]^2$.

30. What is the square of $-\dfrac{5a^3x^2}{7b^2c}$?

SOLUTION

$$\left(-\frac{5a^3x^2}{7b^2c}\right)^2 = -\frac{5a^3x^2}{7b^2c} \times -\frac{5a^3x^2}{7b^2c} = \frac{25a^6x^4}{49b^4c^2}.$$

To raise a fraction to any power:

RULE. — *Raise both numerator and denominator to the required power and prefix the proper sign to the result.*

Raise to the power indicated:

31. $\left(\frac{1}{4b}\right)^2$. 36. $\left(-\frac{5}{ab}\right)^2$. 41. $\left(-\frac{2a}{b}\right)^6$.

32. $\left(\frac{2x}{7y}\right)^2$. 37. $\left(-\frac{2}{3x}\right)^4$. 42. $\left(-\frac{b^3c^2}{a^2x}\right)^2$.

33. $\left(\frac{3x^2}{10y^3}\right)^2$. 38. $\left(-\frac{3x}{2y}\right)^3$. 43. $\left(\frac{a^2b^3}{xy^4}\right)^n$.

34. $\left(\frac{2x^2}{3y}\right)^3$. 39. $\left(-\frac{2a}{x^2y}\right)^5$. 44. $\left(\frac{a^{n-1}b}{x^{2b}y^a}\right)$.

35. $\left(\frac{a^n}{2b^{n-1}}\right)^3$. 40. $\left(-\frac{x^ny^n}{a^2}\right)^7$. 45. $\left(\frac{a^{n-1}c^n}{x^{m+n-1}}\right)^n$.

279. Involution of polynomials.

The following are type forms of *squares* of polynomials:

§ 105, $(a+x)^2 = a^2 + 2ax + x^2$.

§ 108, $(a-x)^2 = a^2 - 2ax + x^2$.

§ 111, $(a - x + y)^2 = a^2 + x^2 + y^2 - 2ax + 2ay - 2xy$.

EXERCISES

280. Raise to the second power:

1. $2a + b$. 5. $3x - 4y^3$. 9. $a - b + x - y$.

2. $2a - b$. 6. $5m^4 - 11$. 10. $a^m + x^n - y^{n+1}$.

3. $a^m - 3b^n$. 7. $1 - 3abc$. 11. $2a + 3b - 4c$.

4. $a^2 - 2x^{2n}$. 8. $4x^4 + 5$. 12. $5a^2 - 1 + 4n^3$.

Raise to the required power by multiplication:

13. $(x+y)^3$. 15. $(x+y)^4$. 17. $(x+y)^5$.

14. $(x-y)^3$. 16. $(x-y)^4$. 18. $(x-y)^5$.

l. Involution of binomials by the Binomial Theorem (§ 549).

actual multiplication,

$$(a+x)^3 = a^3 + 3\,a^2x + 3\,ax^2 + x^3.$$

$$(a-x)^3 = a^3 - 3\,a^2x + 3\,ax^2 - x^3.$$

$$(a+x)^4 = a^4 + 4\,a^3x + 6\,a^2x^2 + 4\,ax^3 + x^4.$$

$$(a-x)^4 = a^4 - 4\,a^3x + 6\,a^2x^2 - 4\,ax^3 + x^4.$$

$$(a+x)^5 = a^5 + 5\,a^4x + 10\,a^3x^2 + 10\,a^2x^3 + 5\,ax^4 + x^5.$$

$$(a-x)^5 = a^5 - 5\,a^4x + 10\,a^3x^2 - 10\,a^2x^3 + 5\,ax^4 - x^5.$$

om the *expansions* just given the following *observations* be made in regard to any *positive integral power* of any *nial*, *a* standing for the first term and *x* for the second:

The number of terms is one greater than the index of the ired power.

The first term contains a only; the last term x only; all terms contain both a and x.

The exponent of a in the first term is the same as the index e required power and it decreases 1 *in each succeeding term; exponent of x in the second term is* 1, *and it increases* 1 *in succeeding term.*

In each term the sum of the exponents of a and x is equal e index of the required power.

The coefficient of the first term is 1; *the coefficient of the nd term is the same as the index of the required power.*

. *The coefficient of any term may be found by multiplying coefficient of the preceding term by the exponent of a in that m, and dividing this product by the number of the term.*

. *All the terms are positive, if both terms of the binomial are sitive.*

8. *The terms are alternately positive and negative, if the second m of the binomial is negative.*

EXERCISES

282. 1. Find the fifth power of $(b-y)$ by the binomial theorem.

SOLUTION

Letters and exponents,	b^5	b^4y	b^3y^2	b^2y^3	by^4	y^5
Coefficients,	1	5	10	10	5	1
Signs,	$+$	$-$	$+$	$-$	$+$	$-$
Combined,	$b^5 - 5b^4y + 10b^3y^2 - 10b^2y^3 + 5by^4 -$					

Expand:

2. $(m+n)^5$.
3. $(m-n)^5$.
4. $(a-c)^3$.
5. $(a+b)^3$.
6. $(b+d)^4$.
7. $(q-r)^5$.
8. $(c+d)^7$.
9. $(x+y)^8$.
10. $(x-y)^4$.
11. $(c-n)^6$.
12. $(x-a)^7$.
13. $(d-y)^8$.
14. $(b+y)^6$.
15. $(m+n)^4$.
16. $(x+2)^2$.
17. $(a+3)^3$.
18. $(x+4)^3$.
19. $(x+5)^3$.
20. $(x-2)^3$.
21. $(a+bc)^4$.
22. $(ab-c)^4$.
23. $(m-pn)^4$.
24. $(ax-by)^3$.
25. $(ax-by)^5$.

26. Expand $(a-x)^4$; then $(2c^2-5)^4$ by the same method.

SOLUTIONS

$$(a-x)^4 = a^4 - 4a^3x + 6a^2x^2 - 4ax^3 + x^4.$$

$$\begin{aligned}(2c^2-5)^4 &= (2c^2)^4 - 4(2c^2)^3 5 + 6(2c^2)^2 5^2 - 4(2c^2) 5^3 + 5^4 \\ &= 16c^8 - 160c^6 + 600c^4 - 1000c^2 + 625.\end{aligned}$$

27. Expand $(1+x^2)^3$.

SOLUTION

$$\begin{aligned}(1+x^2)^3 &= 1^3 + 3(1)^2(x^2) + 3(1)(x^2)^2 + (x^2)^3 \\ &= 1 + 3x^2 + 3x^4 + x^6.\end{aligned}$$

Test. — When $x = 1$, $(1+x^2)^3 = 8$, and $1 + 3x^2 + 3x^4 + x^6 = 8$; hence, $(1+x^2)^3 = 1 + 3x^2 + 3x^4 + x^6$, and the expansion is correct.

pand, and test results :

$(x+2y)^4$. 32. $(1-3x^2)^4$. 36. $(1-x)^7$.

$(2x-y)^3$. 33. $(5x^2-ab)^3$. 37. $(1-2x)^6$.

$(2x-5)^3$. 34. $(1+a^2b^2)^4$. 38. $(x-\frac{1}{2})^6$.

$(x^2-10)^4$. 35. $(2ax-b)^5$. 39. $(\frac{1}{2}x-\frac{1}{3}y)^4$.

pand:

$\left(2a+\frac{1}{2}\right)^5$. 43. $\left(3a^2+\frac{b}{6}\right)^3$. 46. $\left(\frac{1}{2x}-2x\right)^5$.

$\left(\frac{x}{y}-\frac{y}{x}\right)^4$. 44. $\left(1+\frac{3x}{2}\right)^5$. 47. $\left(\frac{1}{a}-a\right)^6$.

$\left(\frac{x}{y}-\frac{y}{x}\right)^6$. 45. $\left(\frac{3}{5}+\frac{5x}{3}\right)^4$. 48. $\left(x+\frac{1}{x}\right)^7$.

Expand $(a-b-c)^3$.

SOLUTION

$-b-c)^3=(\overline{a-b}-c)^3$, a binomial form

$$=(a-b)^3-3(a-b)^2c+3(a-b)c^2-c^3$$
$$=a^3-3a^2b+3ab^2-b^3-3c(a^2-2ab+b^2)+3ac^2-3bc^2-c^3$$
$$=a^3-3a^2b+3ab^2-b^3-3a^2c+6abc-3b^2c+3ac^2-3bc^2-c^3.$$

Expand $(a+b-c-d)^3$.

GESTION. $(a+b-c-d)^3=(\overline{a+b}-\overline{c+d})^3$, a binomial form.

pand:

$(a+x-y)^3$. 57. $(a+2b-3c)^3$.

$(a-m-n)^3$. 58. $(a+b+x+y)^3$.

$(a-x+y)^3$. 59. $(a+b-x-y)^3$.

$(a-x-y)^3$. 60. $(a-b+x-y)^3$.

$(a+x+2)^3$. 61. $(a-b-x+y)^3$.

$(a-x-2)^3$. 62. $(a-b-x-y)^3$.

Binomial Theorem will be treated more fully in §§ 549–557.

EVOLUTION

283. The process $5^2 = 5 \cdot 5 = 25$ illustrates involution.

The process $\sqrt{25} = \sqrt{5 \cdot 5} = 5$ illustrates **evolution,** which will be defined here as the process of finding a root of a number, or as the *inverse of involution.*

For example, $\sqrt{25} = 5$, for $5^2 = 25$;

$$\sqrt[3]{-8} = -2, \text{ for } (-2)^3 = -8.$$

In general, *the nth root of a is a number of which the nth power is a.*

284. Since $25 = 5^2$ and also $25 = (-5)(-5) = (-5)^2$,

$$\sqrt{25} = +5 \text{ or } -5.$$

The roots may be written together thus: ± 5, read '*plus or minus five.*'

Or they may be written ∓ 5, read '*minus or plus five.*'

Similarly, $\sqrt{36} = \pm 6$, $\sqrt{49} = \pm 7$, $\sqrt{\frac{4}{9}} = \pm \frac{2}{3}$.

Every positive number has two square roots.

285. The square root of -16 is not 4, for $4^2 = +16$; nor -4, for $(-4)^2 = +16$. No number so far included in our number system can be a square root of -16 or of any other negative number.

It would be inconvenient and confusing to regard $\sqrt{a}$ as a number only when a is positive. In order to preserve the generality of the discussion of number, it is necessary, therefore, to admit square roots of negative numbers into our number system. The square roots of -16 are written

$$\sqrt{-16} \text{ and } -\sqrt{-16}.$$

Such numbers are called **imaginary numbers** and, in contrast, numbers that do not involve a square root of a negative number are called **real numbers.**

Having extended the number system, we may now state the principle that *every number has two square roots, one positive and the other negative.*

286. Just as every number has two square roots, so every number has three cube roots, four fourth roots, etc.

For example, the cube roots of 8 are the roots of the equation $x^3=8$, which later will be found to be

$$2, \ -1+\sqrt{-3}, \text{ and } -1-\sqrt{-3}.$$

The present discussion is concerned only with *real* roots.

287. Since $2^3=8$, $\sqrt[3]{8}=2.$

Since $(-2)^3=-8$, $\sqrt[3]{-8}=-2.$

Since $2^4=16$ and $(-2)^4=16$, $\sqrt[4]{16}=\pm 2.$

Since $2^5=32$, $\sqrt[5]{32}=2.$

Since $(-2)^5=-32$, $\sqrt[5]{-32}=-2.$

A root is **odd** or **even** according as its index is odd or even. It follows from the law of signs for involution that:

Law of Signs for Real Roots. — *An odd root of a number has the same sign as the number.*

An even root of a positive number may have either sign.

An even root of a negative number is imaginary.

288. A real root of a number, if it has the same sign as the number itself, is called a **principal root** of the number.

The principal square root of 25 is 5, but not -5. The principal cube root of 8 is 2; of -8 is -2.

289. Axiom 7. — *The same roots of equal numbers are equal.*

Thus, if $x=16$, $\sqrt{x}=4$; if $x=8$, $\sqrt[3]{x}=2$; etc.

290. Since $(2^2)^3 = 2^{2\times 3} = 2^6$, the principal cube root of 2^6 is

$$\sqrt[3]{2^6} = 2^{6 \div 3} = 2^2.$$

Law of Exponents for Evolution. — *The exponent of any root of a number is equal to the exponent of the number divided by the index of the root.*

291. 1. Since $(5\,a)^2 = 5^2a^2 = 25\,a^2$, the principal square root of $25\,a^2$ is

$$\sqrt{25\,a^2} = \sqrt{25} \cdot \sqrt{a^2} = 5\,a.$$

2. Since $\left(\frac{a}{b}\right)^4 = \frac{a^4}{b^4}$, the principal fourth root of $\frac{a^4}{b^4}$ is

$$\sqrt[4]{\frac{a^4}{b^4}} = \frac{\sqrt[4]{a^4}}{\sqrt[4]{b^4}} = \frac{a}{b}.$$

Distributive Law for Evolution. — *Any root of a product may be obtained by taking the root of each of the factors and finding the product of the results.*

Any root of the quotient of two numbers is equal to the root of the dividend divided by the root of the divisor.

292. Evolution of monomials.

EXERCISES

1. Extract the square root of $36\,a^6b^2$.

Solution. — Since, in squaring a monomial, § 278, the coefficient is squared and the exponents of the letters are multiplied by 2, to extract the square root, the *square root* of the coefficient must be found, and to it must be annexed the letters each with its exponent *divided* by 2.

The square root of 36 is 6, and the square root of the literal factors is a^3b. Therefore, the *principal* square root of $36\,a^6b^2$ is $6\,a^3b$.

The square root may also be $-6\,a^3b$, since $-6\,a^3b \times -6\,a^3b = 36\,a^6b^2$.

$$\therefore \sqrt{36\,a^6b^2} = \pm 6\,a^3b.$$

2. Extract the cube root of $-125\,x^6y^{21}$.

Solution. $\sqrt[3]{-125\,x^6y^{21}} = -5\,x^2y^7$, the real root.

To find the root of an integral term:

RULE. — *Extract the required root of the numerical coefficient, annex to it the letters each with its exponent divided by the index of the root sought, and prefix the proper sign to the result.*

Find real roots:

3. $\sqrt[3]{a^3b^9c^{15}}$.
4. $\sqrt{a^6b^{16}c^{14}}$.
5. $\sqrt[5]{a^{10}x^5y^{20}}$.
6. $\sqrt[4]{a^{4n}b^8c^{12}}$.
7. $\sqrt{x^{4n}y^8z^{2m}}$.
8. $\sqrt[3]{-8a^6b^{15}}$.
9. $\sqrt[5]{-32x^{10}y^{10}}$.
10. $\sqrt{16x^4y^2}$.
11. $\sqrt[7]{-a^{21}b^{35}x^{14}}$.
12. $\sqrt[5]{-243y^{5n}}$.
13. $\sqrt{(-mb^3)^2}$.
14. $\sqrt[3]{(-a^2b)^9}$.
15. $-\sqrt[9]{a^{18}b^9c^{27}}$.
16. $-\sqrt[3]{-27p^9r^3}$.
17. $-\sqrt[7]{-128a^{14}n^{28}}$.

18. Extract the cube root of $\dfrac{-8x^9y^6}{27m^3n^{12}}$.

SOLUTION. $$\sqrt[3]{\frac{-8x^9y^6}{27m^3n^{12}}} = \frac{\sqrt[3]{-8x^9y^6}}{\sqrt[3]{27m^3n^{12}}} = \frac{-2x^3y^2}{3mn^4} = -\frac{2x^3y^2}{3mn^4}.$$

To find the root of a fractional term:

RULE. — *Find the required root of both numerator and denominator and prefix the proper sign to the resulting fraction.*

Find real roots:

19. $\sqrt{\dfrac{64a^4b^6}{81m^2n^8}}$.
20. $\sqrt[7]{\dfrac{(x-y)^{14}}{128x^{14}}}$.
21. $\sqrt[5]{\dfrac{-32a^5x^{10}}{243y^{15}}}$.
22. $\sqrt[3]{-125\dfrac{x^3}{y^9}}$.
23. $\sqrt[8]{\dfrac{256x^8}{6561}}$.
24. $\sqrt[3]{-\dfrac{125x^{12}y^{12}}{1728c^3}}$.
25. $\sqrt[3]{\left(-\dfrac{27a^3}{64b^6}\right)^2}$.
26. $\sqrt[2n]{\dfrac{a^{2n^2-2n}b^{4n}}{x^{4n}y^{2n}}}$.
27. $\sqrt[n]{\dfrac{b^{4n}c^nx^{3n}}{2^nx^{2n}y^{3n}}}$.

293. To extract the square root of a polynomial.

EXERCISES

1. Find the process for extracting the square root $a^2 + 2ab + b^2$.

PROCESS

$$\begin{array}{lll} & & a^2 + 2ab + b^2 \,\underline{|\,a+b} \\ & & a^2 \\ \text{Trial divisor,} & 2a & \quad 2ab + b^2 \\ \text{Complete divisor,} & 2a + b & \quad 2ab + b^2 \end{array}$$

EXPLANATION.—Since $a^2 + 2ab + b^2$ is the square of $(a + b)$, we knc that the square root of $a^2 + 2ab + b^2$ is $a + b$.

Since the first term of the root is a, it may be found by taking tl square root of a^2, the first term of the power. On subtracting a^2, there a remainder of $2ab + b^2$.

The second term of the root is known to be b, and that may be foun by dividing the first term of the remainder by twice the part of the roc already found. This divisor is called a *trial* divisor.

Since $2ab + b^2$ is equal to $b(2a + b)$, the complete divisor whicl multiplied by b produces the remainder $2ab + b^2$ is $2a + b$; that is, the complete divisor is found by adding the second term of the root to twice the root already found.

On multiplying the complete divisor by the second term of the root and subtracting, there is no remainder; then, $a + b$ is the required root.

2. Extract the square root of $9x^2 - 30xy + 25y^2$.

PROCESS

$$\begin{array}{lll} & & 9x^2 - 30xy + 25y^2 \,\underline{|\,3x-5y} \\ & & 9x^2 \\ \text{Trial divisor,} & 6x & \quad -30xy + 25y^2 \\ \text{Complete divisor,} & 6x - 5y & \quad -30xy + 25y^2 \end{array}$$

Extract the square root of:

3. $4x^2 + 12x + 9$.

4. $x^2 + 2x + 1$.

5. $1 - 4m + 4m^2$.

6. $c^2 - 12c + 36$.

7. $4x^2 + 4x + 1$.

8. $16 + 24x + 9x^2$.

Since, in squaring $a + b + c$, $a + b$ may be represented by x, and the square of the number by $x^2 + 2xc + c^2$, the square root

nber whose root consists of *more than two terms* may be ed in the same way as in exercise 1, *by considering the lready found as one term.*

ind the square root of $4x^4 + 12x^3 - 3x^2 - 18x + 9$.

PROCESS

$$
\begin{array}{l|l}
 & 4x^4 + 12x^3 - 3x^2 - 18x + 9 \;\underline{|\,2x^2 + 3x - 3} \\
 & \underline{4x^4 \qquad\qquad\qquad\quad} \\
4x^2 & 12x^3 - 3x^2 \\
4x^2 + 3x & \underline{12x^3 + 9x^2} \\
4x^2 + 6x & \quad -12x^2 - 18x + 9 \\
4x^2 + 6x - 3 & \quad \underline{-12x^2 - 18x + 9}
\end{array}
$$

LANATION. — Proceeding as in exercise 2, we find that the first two f the root are $2x^2 + 3x$.

sidering $(2x^2 + 3x)$ as the first term of the root, we find the next the root as we found the second term, by dividing the remainder e the part of the root already found. Hence, the trial divisor is $3x$, and the next term of the root is -3. Annexing this, as before, trial divisor already found, we find that the complete divisor is $3x - 3$. Multiplying this by -3 and subtracting the product from $^2 - 18x + 9$, we have no remainder. Hence, the square root of the r is $2x^2 + 3x - 3$.

LE. — *Arrange the terms of the polynomial with reference to nsecutive powers of some letter.*

ract the square root of the first term, write the result as the term of the root, and subtract its square from the given omial.

ide the first term of the remainder by twice the root already , used as a trial divisor, and the quotient will be the next f the root. Write this result in the root, and annex it to al divisor to form the complete divisor.

ltiply the complete divisor by this term of the root, and sub- he product from the first remainder.

d the next term of the root by dividing the first term of the nder by the first term of the trial divisor.

m the complete divisor as before and continue in this man- til all the terms of the root have been found.

Extract the square root of:

10. $25\,a^2 - 40\,a + 16.$

11. $900\,x^2 + 60\,x + 1.$

12. $x^2 + xy + \frac{1}{4}\,y^2.$

13. $4\,x^4 - 52\,x^2 + 169.$

14. $\frac{4}{9}\,d^6 - \frac{2}{3}\,d^3n^2 + \frac{1}{4}\,n^4.$

15. $(a+b)^2 - 4\,(a+b) + 4$

16. $9\,x^4 - 12\,x^3 + 10\,x^2 - 4\,x + 1.$

17. $x^4 - 6\,x^3y + 13x^2y^2 - 12\,xy^3 + 4\,y^4.$

18. $x^8 + 2\,a^6x^2 - a^4x^4 - 2\,a^2x^6 + a^8.$

19. $25\,x^4 + 4 - 12\,x - 30\,x^3 + 29\,x^2.$

20. $1 - 2\,x + 3\,x^2 - 4\,x^3 + 3\,x^4 - 2\,x^5 + x^6.$

21. $a^4 - 2\,a^2b + 2\,a^2c^2 - 2\,bc^2 + b^2 + c^4.$

22. $4\,a^2 - 12\,ab + 16\,ac + 9\,b^2 + 16\,c^2 - 24\,bc.$

23. $9\,x^2 + 25\,y^2 + 9\,z^2 - 30\,xy + 18\,xz - 30\,yz.$

24. $\dfrac{a^2}{9} + \dfrac{10\,a}{3} + 25.$

25. $\dfrac{25}{4\,n^2} + 15 + 9\,n^2.$

26. $\dfrac{25\,d^2}{16\,r^2} - \dfrac{5\,d}{r} + 4.$

27. $\dfrac{x^4}{64} + \dfrac{x^3}{8} - x + 1.$

28. $x^2 + 2\,x - 1 - \dfrac{2}{x} + \dfrac{1}{x^2}$

29. $x^4 + x^3 + \dfrac{13\,x^2}{20} + \dfrac{x}{5} +$

30. $x^8 + 4\,x^7 - 3\,x^4 - 20\,x^5 - 2\,x^6 + 4 + 4\,x^2 - 16\,x + 32\,x^3.$

31. $\dfrac{4\,x^4}{y^4} - \dfrac{4\,x^3}{y^3} - \dfrac{3\,x^2}{y^2} + \dfrac{2\,x}{y} + 1.$

32. $\dfrac{a^4}{4} + a^3x + \dfrac{4\,a^2x^2}{3} + \dfrac{2\,ax^3}{3} + \dfrac{x^4}{9}.$

33. $\dfrac{a^4}{9} - \dfrac{4\,a^3}{3} + \dfrac{a^2b}{3} + 4\,a^2 - 2\,ab + \dfrac{b^2}{4}.$

34. $\dfrac{4\,m^6}{9} - \dfrac{4\,m^5}{3} + \dfrac{19\,m^4}{15} + \dfrac{3\,m^3}{5} - \dfrac{73\,m^2}{50} + \dfrac{3\,m}{10} + \dfrac{9}{16}.$

35. $r^8 - \frac{4}{5}\,r^7 + \frac{4}{25}\,r^6 + \frac{5}{8}\,r^5 - \frac{20}{3}\,r^4 + \frac{12}{5}\,r^3 + \frac{25}{36}\,r^2 - 5\,r + 9.$

36. **Find** four terms of the square root of $1+x$.

SOLUTION

$$\begin{array}{r|l}
 & 1+x \ \underline{|1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3} \\
 & 1 \\
 & x \\
2+\frac{1}{2}x & x+\frac{1}{4}x^2 \\
 & -\frac{1}{4}x^2 \\
2+x-\frac{1}{4}x^2 & -\frac{1}{4}x^2-\frac{1}{8}x^3+\frac{1}{64}x^4 \\
2+x-\frac{1}{4}x^2+\frac{1}{16}x^3 & \frac{1}{8}x^3-\frac{1}{64}x^4
\end{array}$$

Find the square root of the following to four terms:

37. $1-a$.
38. a^2+1.
39. x^2-1.
40. $4-a$.
41. y^2+3.
42. a^2+2b.

SQUARE ROOT OF ARITHMETICAL NUMBERS

294. Compare the number of digits in the square root of each of the following numbers with the number of digits in the number itself:

NUMBER	ROOT	NUMBER	ROOT	NUMBER	ROOT
1	1	1'00	10	1'00'00	100
25	5	10'24	32	56'25'00	750
81	9	98'01	99	99'80'01	999

From the preceding comparison it may be observed that:

PRINCIPLE. — *If a number is separated into periods of two digits each, beginning at units, its square root will have as many digits as the number has periods.*

The left-hand period may be incomplete, consisting of only one digit.

295. If the number of units expressed by the tens' digit is represented by t and the number of units expressed by the units' digit by u, any number consisting of tens and units will be represented by $t+u$, and its square by $(t+u)^2$, or $t^2+2tu+u^2$.

Since $25=20+5$, $25^2=(20+5)^2=20^2+2(20\times 5)+5^2=625$.

EXERCISES

296. 1. Extract the square root of 3844.

FIRST PROCESS

$$\begin{array}{r|l|l} & 38'44 & \underline{60+2} \\ t^2 = & 36\,00 & \\ \hline 2t = 120 & 2\,44 & \\ u = 2 & & \\ \hline 2t+u = 122 & 2\,44 & \end{array}$$

EXPLANATION. — Separating the number into periods of two digits each (Prin., § 294), we find that the root is composed of two digits, tens and units. Since the largest square in 38 is 6, the tens of the root cannot be greater than 6 tens, or 60. Writing 6 tens in the root, squaring, and subtracting from 3844, we have a remainder of 244.

Since the square of a number composed of tens and units is equal to (*the square of the tens*) + (*twice the product of the tens and the units*) + (*the square of the units*), when the square of the tens has been subtracted, the remainder, 244, is twice the product of the tens and the units, plus the square of the units, or only a little more than twice the product of the tens and the units.

Therefore, 244 divided by twice the tens is approximately equal to the units. 2×6 tens, or 120, then, is a *trial*, or *partial*, ***divisor***. On dividing 244 by the trial divisor, the units' figure is found to be 2.

Since twice the tens are to be multiplied by the units, and the units also are to be multiplied by the units to obtain the square of the units, in order to abridge the process the tens and units are first added, forming the *complete divisor* 122, and then multiplied by the units. Thus, $(120 + 2)$ multiplied by $2 = 244$.

Therefore, the square root of 3844 is 62.

SECOND PROCESS

$$\begin{array}{r|l|l} & 38'44 & \underline{62} \\ t^2 = & 36 & \\ \hline 2t = 120 & 2\,44 & \\ u = 2 & & \\ \hline 2t+u = 122 & 2\,44 & \end{array}$$

EXPLANATION. — In practice it is usual to place the figures of the same order in the same column, and to disregard the ciphers on the right of the products.

Since any number may be regarded as composed of tens and *units*, the foregoing processes have a general application.

Thus, $346 = 34$ tens $+ 6$ units; $2377 = 237$ tens $+ 7$ units.

2. Extract the square root of 104976.

Solution

		10'49'76 | 324
		9
Trial divisor	$= 2 \times 30 = 60$	1 49
Complete divisor	$= 60 + 2 = 62$	1 24
Trial divisor	$= 2 \times 320 = 640$	25 76
Complete divisor	$= 640 + 4 = 644$	25 76

RULE. — *Separate the number into periods of two figures each, beginning at units.*

Find the greatest square in the left-hand period and write its root for the first figure of the required root.

Square this root, subtract the result from the left-hand period, and annex to the remainder the next period for a new dividend.

Double the root already found, with a cipher annexed, for a trial divisor, and by it divide the dividend. The quotient, or quotient diminished, will be the second figure of the root. Add to the trial divisor the figure last found, multiply this complete divisor by the figure of the root last found, subtract the product from the dividend, and to the remainder annex the next period for the next dividend.

Proceed in this manner until all the periods have been used. The result will be the square root sought.

1. When the number is not a perfect square, annex periods of decimal ciphers and continue the process.

2. Decimals are pointed off from the decimal point toward the right.

3. The square root of a common fraction may be found by extracting the square root of both numerator and denominator separately or by reducing the fraction to a decimal and then extracting the root.

Extract the square root of:

3. 529.	6. 57121.	9. 2480.04.
4. 2209.	7. 42025.	10. 10.9561.
5. 4761.	8. 95481.	11. .001225.

12. 186624. 13. 1332.25. 14. 111.0916.

15. $\frac{625}{729}$. 17. $\frac{169}{225}$. 19. $\frac{289}{324}$. 21. $\frac{5}{7}$ $\frac{6}{4}$.

16. $\frac{576}{841}$. 18. $\frac{196}{1156}$. 20. $\frac{361}{400}$. 22. $\frac{2}{9}$ $\frac{9}{1}$.

Extract the square root to four decimal places:

23. $\frac{3}{4}$. 25. $\frac{5}{8}$. 27. $\frac{5}{6}$. 29. $\frac{7}{8}$.

24. $\frac{4}{5}$. 26. .6. 28. $\frac{2}{9}$. 30. $\frac{5}{18}$.

297. To extract the cube root of a polynomial.

EXERCISES

1. Find the process for extracting the cube root of $a^3 + 3a^2b + 3ab^2 + b^3$.

PROCESS

$$\begin{array}{lll} & & a^3 + 3a^2b + 3ab^2 + b^3 \,\underline{|a+b} \\ & & a^3 \\ \text{Trial divisor,} & 3a^2 & \overline{\quad 3a^2b + 3ab^2 + b^3} \\ \text{Complete divisor,} & 3a^2 + 3ab + b^2 & \quad 3a^2b + 3ab^2 + b^3 \end{array}$$

EXPLANATION. — Since $a^3 + 3a^2b + 3ab^2 + b^3$ is the cube of $(a + b)$, we know that the cube root of $a^3 + 3a^2b + 3ab^2 + b^3$ is $a + b$.

Since the first term of the root is a, it may be found by taking the cube root of a^3, the first term of the power. On subtracting, there is a remainder of $3a^2b + 3ab^2 + b^3$.

The second term of the root is known to be b, and that may be found by dividing the first term of the remainder by 3 times the square of the part of the root already found. This divisor is called a *trial* divisor.

Since $3a^2b + 3ab^2 + b^3$ is equal to $b(3a^2 + 3ab + b^2)$, the complete divisor, which multiplied by b produces the remainder $3a^2b + 3ab^2 + b^3$, is $3a^2 + 3ab + b^2$; that is, the complete divisor is found by adding to the trial divisor 3 times the product of the first and second terms of the root and the square of the second term of the root.

On multiplying the complete divisor by the second term of the root, and on subtracting, there is no remainder; then, $a + b$ is the required root.

Since, in cubing $a + b + c$, $a + b$ may be expressed by x, the cube of the number will be $x^3 + 3x^2c + 3xc^2 + c^3$. Hence, it is obvious that the cube root of an expression whose root consists *of more than two terms* may be extracted in the same way as in *exercise 1, by considering* the terms already found as one term.

. Find the cube root of $b^6 - 3b^5 + 5b^3 - 3b - 1$.

PROCESS

		$b^6 - 3b^5 + 5b^3 - 3b - 1$	$b^2 - b - 1$
		b^6	
ial divisor,	$3b^4$	$-3b^5 + 5b^3$	
mplete divisor,	$3b^4 - 3b^3 + b^2$	$-3b^5 + 3b^4 - b^3$	
ial divisor,	$3b^4 - 6b^3 + 3b^2$	$-3b^4 + 6b^3 - 3b - 1$	
mplete divisor,	$3b^4 - 6b^3 + 3b + 1$	$-3b^4 + 6b^3 - 3b - 1$	

EXPLANATION. — The first two terms are found in the same manner s in the previous exercise. In finding the next term, $b^2 - b$ is conidered as one term, which we square and multiply by 3 for a trial livisor. On dividing the remainder by this trial divisor, the next term of the root is found to be -1. Adding to the trial divisor 3 times $(b^2 - b)$ multiplied by -1, and the square of -1, we obtain the complete divisor. On multiplying this by -1, and on subtracting the product from $-3b^4 + 6b^3 - 3b - 1$, there is no remainder. Hence, the cube root of the polynomial is $b^2 - b - 1$.

RULE. — *Arrange the polynomial with reference to the consecutive powers of some letter.*

Extract the cube root of the first term, write the result as the first term of the root, and subtract its cube from the given polynomial.

Divide the first term of the remainder by three times the square of the root already found, used as a trial divisor, and the quotient will be the next term of the root.

Add to this trial divisor three times the product of the first and second terms of the root, and the square of the second term. The result will be the complete divisor.

Multiply the complete divisor by the last term of the root found, and subtract this product from the dividend.

Find the next term of the root by dividing the first term of the remainder by the first term of the trial divisor.

Form the complete divisor as before, considering the part of the root already found as the first term, and continue in this manner *until all the terms of the root are found.*

Find the cube root of:

3. $x^3 - 3x^2y + 3xy^2 - y^3$.

4. $m^3 - 9m^2 + 27m - 27$.

5. $8m^3 - 60m^2n + 150mn^2 - 125n^3$.

6. $27x^3 - 189x^2y + 441xy^2 - 343y^3$.

7. $125a^3 + 675a^2x + 1215ax^2 + 729x^3$.

8. $1000p^6 - 300p^4q + 30p^2q^2 - q^3$.

9. $m^6 + 6m^5 + 15m^4 + 20m^3 + 15m^2 + 6m + 1$.

10. $x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$.

11. $x^6 + 3x^5 + 9x^4 + 13x^3 + 18x^2 + 12x + 8$.

12. $x^6 + 12x^5 + 63x^4 + 184x^3 + 315x^2 + 300x + 125$.

13. $x^6 + 6x^5 - 18x^4 - 1000 + 180x^2 - 112x^3 + 600x$.

14. $1 - 6a + 21a^2 - 44a^3 + 63a^4 - 54a^5 + 27a^6$.

15. $8n^3 + 42n^5 - 9n^6 + 36n^4 + 9n^8 - 21n^7 - n^9$.

16. $x^3 - 12x^2 + 54x - 112 + \frac{108}{x} - \frac{48}{x^2} + \frac{8}{x^3}$.

17. $\frac{a^3b^3x^9}{c^3} - \frac{c^3x^6}{b^3} + \frac{3acx^7}{b} - \frac{3a^2bx^8}{c}$.

18. $x^6 + 15x^2 + \frac{15}{x^2} + 20 + \frac{6}{x^4} + \frac{1}{x^6} + 6x^4$.

19. $\frac{1}{x^3} - \frac{3}{2x^2} + \frac{27}{4x} - \frac{49}{8} + \frac{27x}{2} - 6x^2 + 8x^3$.

20. $\frac{n^3}{8} - 3n^2 - \frac{48}{n^2} - 88 + \frac{102}{n} + \frac{51n}{2} + \frac{8}{n^3}$.

21. $c^6 - 3c^5d - 3c^4d^2 + 11c^3d^3 + 6c^2d^4 - 12cd^5 - 8d^6$.

. $\frac{r^3}{s^3}-\frac{9r^2}{s^2}+\frac{30r}{s}-45+\frac{30s}{r}-\frac{9s^2}{r^2}+\frac{s^3}{r^3}.$

. $27x^6-9x^4+55x^2-\frac{325}{27}+\frac{110}{3x^2}-\frac{4}{x^4}+\frac{8}{x^6}.$

. $\frac{8a^9}{27}+2a^7-\frac{4a^6}{3}+\frac{9a^5}{2}-6a^4+\frac{43a^3}{8}-\frac{27a^2}{4}+\frac{9a}{2}-1.$

. $\frac{b^9}{a^6}+\frac{3b^8}{a^7}+\frac{6b^7}{a^8}+\frac{7b^6}{a^9}+\frac{6b^5}{a^{10}}+\frac{3b^4}{a^{11}}+\frac{b^3}{a^{12}}.$

. $n^6-\frac{3}{2}n^5+\frac{9}{4}n^4-\frac{13}{8}n^3+\frac{9}{8}n^2-\frac{3}{8}n+\frac{1}{8}.$

. $\frac{1}{27}r^6-\frac{1}{6}r^5-\frac{3}{4}r^4+\frac{23}{8}r^3+\frac{27}{4}r^2-\frac{27}{2}r-27.$

. $\frac{1}{8}x^6+\frac{3}{4}x^5y+x^4y^2-x^3y^3-\frac{4}{3}x^2y^4+\frac{4}{3}xy^5-\frac{8}{27}y^6.$

CUBE ROOT OF ARITHMETICAL NUMBERS

Compare the number of digits in each number and in root:

ROOT	NUMBER	ROOT	NUMBER	ROOT
1	1'000	10	1'000'000	100
3	27'000	30	27'000'000	300
9	970'299	99	997'002'999	999

ve that:

CIPLE. — *If a number is separated into periods of three ch, beginning at units, its cube root will have as many the number has periods.*

ft-hand period may be incomplete, consisting of only one or two

If the number of units expressed by the tens' digit is ated by t, and the number of units expressed by the igit is represented by u, any number consisting of tens ts will be represented by $t+u$, and its cube by $(t+u)^3$, $3t^2u+3tu^2+u^3$.

$25 = 2$ tens $+ 5$ units, or $(20+5)$ units,

$25^3 = 20^3 + 3(20^2 \times 5) + 3(20 \times 5^2) + 5^3 = 15625.$

EXERCISES

300. 1. **Extract the cube root of 12167.**

FIRST PROCESS

$$\begin{array}{rrr|l}
 & & 12'167 & \underline{20+3} \\
 & t^3 = & 8\,000 & \\
\cline{2-3}
\text{Trial divisor,} & 3\,t^2 = 1200 & \multicolumn{1}{|l}{4\,167} & \\
 & 3\,tu = 180 & \multicolumn{1}{|l}{} & \\
 & u^2 = 9 & \multicolumn{1}{|l}{} & \\
\cline{2-2}
\text{Complete divisor,} & = 1389 & \multicolumn{1}{|l}{\underline{4\,167}} &
\end{array}$$

EXPLANATION. — On separating 12167 into periods of three fig (§ 298, Prin.) there are found to be two digits in the root, th root is composed of *tens* and *units*. Since the cube of tens is tl and the thousands of the power are less than 27, or 3^3, and moı or 2^3, the tens' figure of the root is 2. 2 tens, or 20, cubed is 8000, subtracted from 12167 leaves 4167, which is equal to 3 times th the units + 3 times the tens × the units² + the units³.

Since 3 times the tens² × the units is much greater than 3 tens × the units² + the units³, 4167 is only a little more than 3 tens² × the units. If, then, 4167 is divided by 3 times the ter 1200, the trial divisor, the quotient will be approximately equ units, that is, 3 will be the units of the root, provided proper has been made for the additions necessary to obtain the complet

Since the complete divisor is found by adding to 3 times the sum of 3 times the tens × the units and the units², the divisor is 1200 + 180 + 9, or 1389. This multiplied by 3, the uı 4167, which, subtracted from 4167, leaves no remainder.

Therefore, the cube root of 12167 is 20 + 3, or 23.

SECOND PROCESS

$$\begin{array}{rr|l}
 & 12'167 & \underline{23} \\
t^2 \;= & 8 & \\
\hline
3\,t^2 \;= 1200 & \multicolumn{1}{|l}{4\,167} & \\
3\,tu = 180 & \multicolumn{1}{|l}{} & \\
u^2 = 9 & \multicolumn{1}{|l}{} & \\
\cline{1-1}
1389 & \multicolumn{1}{|l}{\underline{4\,167}} &
\end{array}$$

EXPLANATION. — In practice it i place figures of the same order in column, and to disregard the ciphe right of the products.

Since a root expressed by a ber of figures may be rega composed of tens and units, *cesses of* exercise 1 have a general application.

Thus, 120 = 12 tens + 0 units; 1203 = 120 tens + 3 units.

2. Extract the cube root of 1740992427.

SOLUTION

				1'740'992'427 | 1203
	$t^3 =$			1
Complete divisor	$3t^2 = 3(10)^2$	=	300	740
	$3tu = 3(10 \times 2)$	=	60	
	$u^2 = 2^2$	=	4	
			364	728
				12992
	$3t^2 = 3(120)^2$	=	43200	
Complete divisor	$3t^2 = 3(1200)^2$	=	4320000	12992427
	$3tu = 3(1200 \times 3)$	=	10800	
	$u^2 = 3^2$	=	9	
			4330809	12992427

Since the third figure of the root is 0, it is not necessary to form the complete divisor, inasmuch as the product to be subtracted will be 0.

RULE.—*Separate the number into periods of three figures each, beginning at units. Find the greatest cube in the left-hand period, and write its root for the first digit of the required root.*

Cube this root, subtract the result from the left-hand period, and annex to the remainder the next period for a new dividend.

Take three times the square of the root already found, annex two ciphers for a trial divisor, and by the result divide the dividend. The quotient, or quotient diminished, will be the second figure of the root.

To this trial divisor add three times the product of the first part of the root with a cipher annexed, multiplied by the second part, and also the square of the second part. Their sum will be the complete divisor.

Multiply the complete divisor by the second part of the root, and subtract the product from the dividend.

Continue thus until all the figures of the root have been found.

1. When there is a remainder after subtracting the last product, annex decimal ciphers, and continue the process.

2. Decimals are pointed off from the decimal point toward the right.

3. The cube root of a common fraction may be found by extracting the cube root of the numerator and the denominator separately or by reducing the fraction to a decimal and then extracting its root.

Extract the cube root of:

3. 29791.
4. 54872.
5. 110592.
6. 300763.
7. 681472.
8. 941192.
9. 2406104.
10. 69426531.
11. 28372625.
12. 48.228544.
13. 17173.512.
14. 95.443993.
15. .000024389.
16. .001906624.
17. .000912673.
18. .259694072.
19. 926.859375.
20. 514500.05819

Extract the cube root to three decimal places:

21. 2.
22. 5.
23. .8.
24. .16.
25. $\frac{5}{64}$.
26. $\frac{2}{3}$.
27. $\frac{7}{8}$.
28. $\frac{3}{16}$

ROOTS BY VARIOUS METHODS

301. By inspection and trial.

To find the cube root of a number, as 343, we estimate t root and cube it. If the cube is greater or less than the nu ber, our estimate must be modified, for the cube of the r must be the number itself.

This method, which is the general one in evolution, may used to find any root of a polynomial.

By inspection we estimate $\sqrt[5]{x^5 - 10x^4 + 40x^3 - 80x^2 + 80x -}$ be $x - 2$, noting the number of terms and the first and last te By trial $x - 2$ proves to be the root, for its fifth power is found the given polynomial.

302. By factoring.

This method consists in factoring, grouping the factor extracting the root of each group.

Thus, $\sqrt[3]{42875} = \sqrt[3]{5 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 7} = \sqrt[3]{5^3 \cdot 7^3} = 5 \cdot 7 = 35;$

$$\sqrt{\quad x^3 - 3x^2 - 4x + 4} = \sqrt{(x-1)^2(x+2)^2} = (x - 1$$

3. By successive extraction of roots.

nce the *fourth* power is the *square* of the *second* power, the power the *cube* of the *second* power, etc., any indicated whose index is 4, 6, 8, 9, etc., may be found by extracting essively the roots corresponding to the factors of the index.

e fourth root may be obtained by extracting the square root of the re root; the sixth root, by extracting the cube root of the square root, e square root of the cube root; the eighth root, by extracting the re root of the square root of the square root.

EXERCISES

4. Using any method, find the:

. Square root of $a^6 - 12a^3 + 36$.

. Cube root of $125 - 75x + 15x^2 - x^3$.

. Fourth root of $16 - 32x + 24x^2 - 8x^3 + x^4$.

. Fourth root of $x^4 + 12x^3y + 54x^2y^2 + 108xy^3 + 81y^4$.

. Fourth root of $16m^4 - 32m^3 + 24m^2 - 8m + 1$.

. Fifth root of $32x^5 + 80x^4 + 80x^3 + 40x^2 + 10x + 1$.

. Fifth root of $a^{10} + 15a^8 + 90a^6 + 270a^4 + 405a^2 + 243$.

ind the sixth root of:

8. $x^6 - 12x^5 + 60x^4 - 160x^3 + 240x^2 - 192x + 64$.

9. $64x^6 - 576x^5 + 2160x^4 - 4320x^3 + 4860x^2 - 2916x + 729$.

0. $x^6 + 6acx^5 + 15a^2c^2x^4 + 20a^3c^3x^3 + 15a^4c^4x^2 + 6a^5c^5x + a^6c^6$.

ind the indicated root:

1. $\sqrt[3]{3375}$.
2. $\sqrt[4]{1296}$.
3. $\sqrt[4]{50625}$.
4. $\sqrt[6]{46656}$.

15. $\sqrt[6]{262144}$.
16. $\sqrt[5]{759375}$.
17. $\sqrt[6]{531441}$.
18. $\sqrt[8]{5764801}$.

19. $\sqrt[5]{4084101}$.
20. $\sqrt[8]{16777216}$.
21. $\sqrt[6]{24137569}$.
22. $\sqrt[9]{10604499373}$.

THEORY OF EXPONENTS

305. Thus far the exponents used have been *positive integers* only, and consequently the **laws of exponents** have been obtained in the following restricted forms:

1. $a^m \times a^n = a^{m+n}$ when m and n are positive integers.

2. $a^m \div a^n = a^{m-n}$ when m and n are positive integers and m is greater than n.

3. $(a^m)^n = a^{mn}$ when m and n are positive integers.

4. $\sqrt[n]{a^m} = a^{m \div n}$ when m and n are positive integers, and m is a multiple of n.

5. $(ab)^n = a^n b^n$ when n is a positive integer.

If all restrictions are removed from m and n, we may then have expressions like a^{-2} and $a^{\frac{3}{5}}$. But such expressions are as yet unintelligible, because -2 and $\frac{3}{5}$ cannot show how many times a number is used as a factor.

Since, however, these forms may occur in algebraic processes, it is important to discover meanings for them that will allow their use in accordance with the laws already established, for otherwise great complexity and confusion would arise in the processes involving them.

Assuming that the law of exponents for multiplication,

$$a^m \times a^n = a^{m+n},$$

is true for *all* values of m and n, the meanings of *zero, negative,* and *fractional* exponents may be readily discovered by substituting these different kinds of exponents for m and n or both, *and observing* to what conclusions we are led.

306. Meaning of a zero exponent.

We have agreed that any new kind of exponent shall have a meaning determined in harmony with the law of exponents for multiplication, expressed by the formula,

$$a^m \times a^n = a^{m+n}.$$

If $n = 0$, $\qquad a^m \times a^0 = a^{m+0}$, or a^m.

Dividing by a^m, Ax. 4, $\qquad a^0 = \dfrac{a^m}{a^m} = 1$. That is,

Any number with a zero exponent is equal to 1.

307. Meaning of a negative exponent.

Since, § 305, $a^m \times a^n = a^{m+n}$, is to hold true for all values of m and n, if $m = -n$,

$$a^{-n} \times a^n = a^{-n+n} = a^0.$$

But, § 306, $\qquad a^0 = 1.$

Hence, Ax. 5, $\qquad a^{-n} \times a^n = 1.$

Dividing by a^n, Ax. 4, $\qquad a^{-n} = \dfrac{1}{a^n}$. That is,

Any number with a negative exponent is equal to the reciprocal of the same number with a numerically equal positive exponent.

308. By the definition of negative exponent just given,

$$a^{-m} = \frac{1}{a^m} \text{ and } b^{-n} = \frac{1}{b^n}.$$

Therefore, $$\frac{a^{-m}}{b^{-n}} = \frac{\frac{1}{a^m}}{\frac{1}{b^n}} = \frac{1}{a^m} \times \frac{b^n}{1} = \frac{b^n}{a^m}.$$ Hence,

PRINCIPLE. — *Any factor may be transferred from one term of a fraction to the other without changing the value of the fraction, provided the sign of the exponent is changed.*

EXERCISES

309. Find a simple value for:

1. 5^0. 3. 2^{-5}. 5. $(-3)^0$. 7. $(a^n b$

2. 4^{-2}. 4. 3^{-3}. 6. $(-6)^{-2}$. 8. $(-\frac{1}{3}$

9. Which is the greater, $(\frac{1}{5})^2$ or $(\frac{1}{5})^3$? $(\frac{1}{5})^{-2}$ or $(\frac{1}{5})^{-3}$?

10. Find the value of $2^3 - 3 \cdot 2^2 + 5 \cdot 2^1 - 7 \cdot 2^0 + 4 \cdot 2^{-1} -$

11. Find the value of $x^2 - 3x^1 + 4x^0 + x^{-1} - 5x^{-2} + x^{-3}$ wh $x = \frac{1}{2}$; when $x = -\frac{1}{2}$; when $x = 1$.

12. Which is the greater, $(-\frac{1}{3})^{-3}$ or $(\frac{1}{3})^3$? $(-\frac{1}{3})^{-4}$ or $(\frac{1}{3})$

Write with negative exponents:

13. $1 \div 5$. 15. $1 \div 2^n$. 17. $c \div a^2x^3$.

14. $1 \div a^2$. 16. $a \div x^3$. 18. $am^3 \div bx^n$.

19. Write $5x^{-3}y^2$ with positive exponents.

SOLUTION. — By § 307, $5x^{-3}y^2 = 5y^2\frac{1}{x^3} = \frac{5y^2}{x^3}$.

Write with positive exponents:

20. $2x^{-1}$. 23. $a^{-1}b^{-1}$. 26. $4a^2c^{-2}$.

21. $5a^{-5}$. 24. $x^{-3}y^{-2}$. 27. $3ax^{-2}$.

22. $3b^{-2}$. 25. $a^{-1}b^2c^{-3}$. 28. a^nb^{-3n}.

29. $4x^3 - 2x^2 + 5x^1 - 6x^0 + 3x^{-1} - 5x^{-2}$.

30. $2a^3 - 12a^2 - 16a + 3a^0 + 2a^{-1} - 7a^{-2}$.

31. $a^3b^{-3} - a^2b^{-2} + ab^{-1} - 1 + a^{-1}b - a^{-2}b^2 + a^{-3}b^3$.

32. Write $\frac{3a^2y}{bx^2}$ without a denominator.

SOLUTION. — By § 308, $\frac{3a^2y}{bx^2} = 3a^2b^{-1}x^{-2}y$.

Write without a denominator:

33. $\frac{ax}{by}$. 34. $\frac{mn}{a^2}$. 35. $\frac{1}{a^{-2}b^2}$. 36. $\left(\frac{x}{y}\right)^2$.

$\frac{c^2d^2}{a^2b^2}$. 40. $\frac{1}{m^{-3}n}$. 43. $\frac{4}{x^{-4}}$. 46. $\left(\frac{a}{b^2}\right)^3$.

$\frac{a^2m^3}{b^3n^2}$. 41. $\frac{3a^2c^{-2}}{x^2}$. 44. $\frac{x^{-1}}{y}$. 47. $\frac{a^{-3}}{b^{-3}}$.

$\frac{b}{x^{-6}}$. 42. $\frac{2a^2x^{-3}}{b^{-4}y}$. 45. $\left(\frac{3}{m}\right)^3$. 48. $\frac{1}{(ab)^3}$.

. Meaning of a fractional exponent.

ce (§ 305) the first law of exponents is to hold true for ponents,

$$a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a;$$

is, $a^{\frac{1}{2}}$ is one of the two equal factors of a, or is a square of a. The other square root of a is $-a^{\frac{1}{2}}$.

ain,

$$a^{\frac{3}{2}} \times a^{\frac{3}{2}} = a^{\frac{3}{2}+\frac{3}{2}} = a^3;$$

is, $a^{\frac{3}{2}}$ is a square root of the cube of a.

, similarly, $a^{\frac{1}{2}} \times a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}+\frac{1}{2}} = a^{\frac{3}{2}}$;

is, $a^{\frac{3}{2}}$ is the cube of a square root of a.

general, confining the discussion to principal roots, let nd q be any two positive integers. By the first law of nents, § 305, $a^{\frac{p}{q}} \cdot a^{\frac{p}{q}} \cdots$ to q factors $= a^{\frac{p}{q}+\frac{p}{q}+\cdots \text{ to } q \text{ terms}} = a^p$.

herefore $a^{\frac{p}{q}}$, one of the q equal factors of a^p, is a qth root he pth power of a.

imilarly, $a^{\frac{1}{q}}$ is a qth root of a.

lso, since $a^{\frac{1}{q}} \cdot a^{\frac{1}{q}} \cdots$ to p factors $= a^{\frac{1}{q}+\frac{1}{q}+\cdots \text{ to } p \text{ terms}} = a^{\frac{p}{q}}$, $a^{\frac{p}{q}}$ is pth power of a qth root of a.

The numerator of a fractional exponent with positive integral ms indicates a power and the denominator a root.

The fraction as a whole indicates a root of a power or a power root.

311. Any fractional exponent that does not itself involve a root sign may be reduced to one of the forms $\frac{p}{q}$ or $-\frac{p}{q}$.

Thus, $$8^{-\frac{\frac{8}{3}}{2}} = 8^{-\frac{4}{3}}.$$

By §§ 307, 291, $$a^{-\frac{p}{q}} = \frac{1}{a^{\frac{p}{q}}} = \frac{1^{\frac{p}{q}}}{a^{\frac{p}{q}}} = \left(\frac{1}{a}\right)^{\frac{p}{q}}.$$

EXERCISES

312. 1. Find the value of $16^{\frac{3}{4}}$.

FIRST SOLUTION.
$$\begin{aligned} 16^{\frac{3}{4}} &= \sqrt[4]{16^3} = \sqrt[4]{16 \cdot 16 \cdot 16} \\ &= \sqrt[4]{(2 \cdot 2 \cdot 2 \cdot 2)(2 \cdot 2 \cdot 2 \cdot 2)(2 \cdot 2 \cdot 2 \cdot 2)} \\ &= \sqrt[4]{(2 \cdot 2 \cdot 2)(2 \cdot 2 \cdot 2)(2 \cdot 2 \cdot 2)(2 \cdot 2 \cdot 2)} \\ &= 2 \cdot 2 \cdot 2 = 8. \end{aligned}$$

SECOND SOLUTION. $16^{\frac{3}{4}} = (16^{\frac{1}{4}})^3 = 2^3 = 8.$

In numerical exercises it is usually best to extract the root first.

Simplify, taking only principal roots:

2. $8^{\frac{1}{3}}$.
3. $8^{\frac{2}{3}}$.
4. $8^{-\frac{1}{3}}$.
5. $(-8)^{\frac{1}{3}}$.
6. $64^{\frac{2}{3}}$.
7. $32^{\frac{3}{5}}$.
8. $25^{\frac{3}{2}}$.
9. $81^{\frac{3}{4}}$.
10. $64^{-\frac{2}{3}}$.
11. $(-8)^{-\frac{4}{3}}$.
12. $(-32)^{-\frac{3}{5}}$.
13. $16^{-\frac{3}{4}}$.

14. Which is the greater, $27^{\frac{2}{3}}$ or $(-27)^{-\frac{2}{3}}$?

15. Which is the greater, $(\frac{1}{4})^{\frac{3}{2}}$ or $(\frac{1}{4})^{-\frac{3}{2}}$?

16. Which is the greater, $64^{-\frac{2}{3}}$ or $(\frac{1}{64})^{\frac{3}{2}}$? $64^{\frac{2}{3}}$ or $(\frac{1}{64})^{-\frac{3}{2}}$?

17. Find the value of $x^{-\frac{4}{3}} - 4x^{-\frac{2}{3}} + 4$ when $x = -\frac{1}{125}$.

18. Express $\sqrt[3]{a^2bc^{-4}}$ with positive fractional exponents.

SOLUTION. $$\sqrt[3]{a^2bc^{-4}} = a^{\frac{2}{3}}b^{\frac{1}{3}}c^{-\frac{4}{3}} = \frac{a^{\frac{2}{3}}b^{\frac{1}{3}}}{c^{\frac{4}{3}}}.$$

Express with positive fractional exponents:

19. $\sqrt{ab^3}$. **22.** $(\sqrt{x})^3$. **25.** $(\sqrt[3]{xy})^{-2}$.

20. $\sqrt{xy}$. **23.** $(\sqrt[5]{y})^4$. **26.** $5\sqrt{x^{-1}y^{-1}}$.

21. $\sqrt{x^3y^5}$. **24.** $(\sqrt[5]{ab})^3$. **27.** $2\sqrt[3]{(a+b)^2}$.

Express roots with radical signs and powers with positive exponents:

28. $a^{\frac{2}{3}}$. **30.** $x^{\frac{5}{4}}$. **32.** $x^{\frac{5}{6}}y^{\frac{1}{6}}$. **34.** $a^{\frac{1}{2}} \div x^{\frac{1}{2}}$.

29. $x^{\frac{4}{5}}$. **31.** $a^{\frac{1}{3}}b^{\frac{2}{3}}$. **33.** $a^{\frac{1}{4}}b^{-\frac{3}{4}}$. **35.** $x^{\frac{2}{3}} \div y^{\frac{4}{3}}$.

36. Simplify $\sqrt[3]{x^2} + x^{\frac{1}{3}} + 8^{\frac{2}{3}} + 3x^{\frac{2}{3}} - 5\sqrt[3]{x} - \sqrt[3]{27^2}$.

37. Simplify $4\sqrt[5]{x} + 5x^0 - 3x^{-\frac{1}{5}} + 2\sqrt[5]{x^{-1}} - 8^{\frac{2}{3}} - 2x^{\frac{1}{5}}$.

313. Operations involving positive, negative, zero, and fractional exponents.

Since zero, negative, and fractional exponents have been defined in conformity with the law of exponents for multiplication, this law holds true for all exponents so far encountered. For the proofs of the generality of the other laws of exponents, see the author's *Academic Algebra*.

EXERCISES

314. Multiply:

1. a^3 by a^{-2}. **3.** a^4 by a^{-4}. **5.** a^2 by a^0.

2. a^2 by a^{-1}. **4.** a by a^{-3}. **6.** $x^{\frac{1}{2}}$ by $x^{\frac{1}{2}}$.

7. $a^{\frac{1}{2}}b^{\frac{1}{3}}$ by $a^{\frac{1}{2}}b^{\frac{2}{3}}$. **10.** n^{-2} by $an^{\frac{5}{2}}$.

8. $m^{\frac{2}{3}}n$ by $m^{\frac{3}{5}}n^{-1}$. **11.** a^{m-n} by a^{n-p}.

9. $a^{\frac{1}{3}}b^{\frac{1}{4}}$ by $a^{-\frac{4}{3}}b^{\frac{3}{4}}$. **12.** $a^{\frac{m+n}{2}}$ by $a^{\frac{m-n}{2}}$.

13. Multiply $x^{\frac{4}{5}}y^{-\frac{1}{5}} + x^{\frac{3}{5}} + x^{\frac{2}{5}}y^{\frac{1}{5}} + x^{\frac{1}{5}}y^{\frac{2}{5}} + y^{\frac{3}{5}}$ by $x^{\frac{1}{5}}y^{\frac{1}{5}}$.

14. Multiply $y^n + x^{-1}y^{n+1} + x^{-2}y^{n+2} + x^{-3}y^{n+3}$ by $x^n y^{-n}$.

15. Expand $(a^{\frac{1}{3}}b^{-\frac{1}{2}}+1+a^{-\frac{1}{3}}b^{\frac{1}{2}})(a^{\frac{1}{3}}b^{-\frac{1}{2}}-1+a^{-\frac{1}{3}}b^{\frac{1}{2}})$.

FIRST SOLUTION

$$\begin{array}{lllll}
a^{\frac{1}{3}}b^{-\frac{1}{2}} & +\ 1 & +\ a^{-\frac{1}{3}}b^{\frac{1}{2}} & & \\
a^{\frac{1}{3}}b^{-\frac{1}{2}} & -\ 1 & +\ a^{-\frac{1}{3}}b^{\frac{1}{2}} & & \\
\hline
a^{\frac{2}{3}}b^{-1} & +\ a^{\frac{1}{3}}b^{-\frac{1}{2}} & +\ a^0b^0 & & \\
 & -\ a^{\frac{1}{3}}b^{-\frac{1}{2}} & -\ 1 & -\ a^{-\frac{1}{3}}b^{\frac{1}{2}} & \\
 & & +\ a^0b^0 & +\ a^{-\frac{1}{3}}b^{\frac{1}{2}} & +\ a^{-\frac{2}{3}}b \\
\hline
a^{\frac{2}{3}}b^{-1} & & +\ 1 & & +\ a^{-\frac{2}{3}}b
\end{array}$$

SECOND SOLUTION

$$(a^{\frac{1}{3}}b^{-\frac{1}{2}}+1+a^{-\frac{1}{3}}b^{\frac{1}{2}})(a^{\frac{1}{3}}b^{-\frac{1}{2}}-1+a^{-\frac{1}{3}}b^{\frac{1}{2}})$$

§ 114, $$=(a^{\frac{1}{3}}b^{-\frac{1}{2}}+a^{-\frac{1}{3}}b^{\frac{1}{2}})^2-1^2$$

$$=(a^{\frac{2}{3}}b^{-1}+2\,a^0b^0+a^{-\frac{2}{3}}b)-1$$

$$=a^{\frac{2}{3}}b^{-1}+2+a^{-\frac{2}{3}}b-1=a^{\frac{2}{3}}b^{-1}+1+a^{-\frac{2}{3}}b.$$

Expand:

16. $(a^{\frac{1}{2}}+b^{\frac{1}{2}})(a^{\frac{1}{2}}-b^{\frac{1}{2}})$.

17. $(x^{\frac{2}{3}}+y^{\frac{2}{3}})(x^{\frac{2}{3}}-y^{\frac{2}{3}})$.

18. $(x^{-\frac{1}{2}}+10)(x^{-\frac{1}{2}}-1)$.

19. $(x^{\frac{2}{5}}-4)(x^{\frac{2}{5}}+5)$.

20. $(x^{\frac{2}{5}}-x^{\frac{1}{5}}y^{-\frac{1}{5}}+y^{-\frac{2}{5}})(x^{\frac{1}{5}}+y^{-\frac{1}{5}})$.

21. $(a^{\frac{2}{3}}+b^{-\frac{2}{3}}+a^{\frac{1}{3}}b^{-\frac{1}{3}}+1)(a^{\frac{1}{3}}-b^{-\frac{1}{3}})$.

22. $(1-x+x^2)(x^{-3}+x^{-2}+x^{-1})$.

23. $(a^{-1}+b^{-\frac{1}{2}}+c^{\frac{1}{2}})(a^{-1}+b^{-\frac{1}{2}}+2\,c^{\frac{1}{2}})$.

Divide:

24. a^5 by a^6.

25. a^3 by a^0.

26. a^2 by a^{-2}.

27. $x^{\frac{5}{2}}$ by $x^{-\frac{1}{2}}$.

28. $x^{\frac{1}{2}}$ by $x^{\frac{1}{3}}$.

29. $x^{n-\frac{3}{2}}$ by x^{n-2}.

30. Divide $x^4+x^2y^2+y^4$ by x^2y^2.

31. Divide $a^{-4}+a^{-2}b+b^2$ by $a^{-2}b$.

32. Divide $x^4+2\,ax^3+3\,a^2x^2+a^3x-a^4$ by a^4x^4.

33. Divide $b^{-1}+3a^{-\frac{1}{2}}-10a^{-1}b$ by $a^{\frac{1}{2}}b^{-1}-2$.

SOLUTION

$$
\begin{array}{l}
a^{\frac{1}{2}}b^{-1}-2)\,b^{-1}+3a^{-\frac{1}{2}}-10a^{-1}b\,(a^{-\frac{1}{2}}+5a^{-1}b \\
\qquad\qquad\underline{b^{-1}-2a^{-\frac{1}{2}}} \\
\qquad\qquad\qquad 5a^{-\frac{1}{2}}-10a^{-1}b \\
\qquad\qquad\qquad \underline{5a^{-\frac{1}{2}}-10a^{-1}b}
\end{array}
$$

Divide:

. $a-b$ by $a^{\frac{1}{2}}+b^{\frac{1}{2}}$.

. $a-b$ by $a^{\frac{1}{3}}-b^{\frac{1}{3}}$.

. $a+b$ by $a^{\frac{1}{3}}+b^{\frac{1}{3}}$.

. a^2+b^2 by $a^{\frac{2}{3}}+b^{\frac{2}{3}}$.

38. $x-1$ by $x^{\frac{2}{3}}+x^{\frac{1}{3}}+1$.

39. $x^{\frac{4}{7}}-2+x^{-\frac{4}{7}}$ by $x^{\frac{2}{7}}-x^{-\frac{2}{7}}$.

40. $3-4x^{-1}+x^{-2}$ by $x^{-1}-3$.

41. a^2-b^3 by $a^{\frac{1}{3}}+b^{\frac{1}{2}}$.

Simplify the following:

. $(a^{\frac{1}{2}})^2$.

. $(a^{-\frac{1}{3}})^6$.

. $(a^{-4})^2$.

. $\sqrt[4]{x^{\frac{2}{3}}}$.

. $\sqrt[4]{a^{-\frac{1}{2}}}$.

. $\sqrt[4]{a^{-\frac{8}{3}}}$.

48. $(-a^{\frac{1}{2}})^3$.

49. $(-a^2)^4$.

50. $(-a^{\frac{4}{5}})^{-1}$.

51. $\sqrt[3]{a^{-\frac{1}{2}}b^{-3}}$.

52. $\sqrt{x^{\frac{4}{3}}y^{-3}}$.

53. $\sqrt[n]{a^xb^y}$.

54. $(8^{-\frac{1}{3}})^4$.

55. $(16^{-\frac{1}{2}})^3$.

56. $(-\frac{1}{32}a^{10})^{-\frac{3}{5}}$.

57. $(\frac{1}{36}a^{-\frac{2}{3}}b^{\frac{1}{3}})^{-\frac{3}{2}}$.

58. $(\frac{1}{9}m^{-1}n^{-\frac{1}{2}})^{\frac{1}{2}}$.

59. $(4x^{2n}y^{-3}z^4)^{\frac{5}{2}}$.

Expand by the binomial formula:

. $(a^{\frac{1}{2}}-b^{\frac{1}{2}})^2$.

. $(a^{\frac{1}{2}}+b^{\frac{1}{3}})^3$.

62. $(a^{-1}-b^{\frac{2}{3}})^3$.

63. $(x^{-\frac{1}{2}}-y^{\frac{1}{2}})^4$.

64. $(a^{-\frac{1}{2}}+\frac{1}{2})^3$.

65. $(1-x^{\frac{3}{2}})^4$.

Extract the square root of:

66. $x^2+2x^{\frac{3}{2}}+3x+4x^{\frac{1}{2}}+3+2x^{-\frac{1}{2}}+x^{-1}$.

67. $x^2+y+4z^{-2}-2xy^{\frac{1}{2}}+4xz^{-1}-4y^{\frac{1}{2}}z^{-1}$.

68. $a+4b^{\frac{2}{3}}+9c^{\frac{1}{2}}-4a^{\frac{1}{2}}b^{\frac{1}{3}}+6a^{\frac{1}{2}}c^{\frac{1}{4}}-12b^{\frac{1}{3}}c^{\frac{1}{4}}$.

Extract the cube root of:

69. $a^2+6a^{\frac{4}{3}}+12a^{\frac{2}{3}}+8.$

70. $a-3a^{\frac{2}{3}}b^{\frac{2}{3}}+3a^{\frac{1}{3}}b^{\frac{4}{3}}-b^2.$

71. $8x^{-1}-12x^{-\frac{2}{3}}y+6x^{-\frac{1}{3}}y^2-y^3.$

72. $x^{\frac{3}{2}}-6x+15x^{\frac{1}{2}}-20+15x^{-\frac{1}{2}}-6x^{-1}+x^{-\frac{3}{2}}.$

73. Factor $4x^{-2}-9y^{-2}$, and express the result with positive exponents.

SOLUTION. — By § 152, $4x^{-2}-9y^{-2}=(2x^{-1}+3y^{-1})(2x^{-1}-3y^{-1})$

$$=\left(\frac{2}{x}+\frac{3}{y}\right)\left(\frac{2}{x}-\frac{3}{y}\right).$$

Factor, expressing results with positive exponents:

74. $a^{-2}-b^{-2}.$

75. $9-x^{-2}.$

76. $16-a^{-4}.$

77. $27-b^{-3}.$

78. $b^{-3}+y^{-3}.$

79. $x^3-x^{-3}.$

80. $a^2+2+a^{-2}.$

81. $b^4-8+16b^{-4}.$

82. $12-x^{-1}-x^{-2}.$

83. $2-3x^{-1}-2x^{-2}.$

Solve for values of x corresponding to principal roots, and test each result:

84. $x^{\frac{1}{2}}=7.$

85. $x^{\frac{3}{4}}=8.$

86. $x^{\frac{2}{3}}=9.$

87. $x^{\frac{4}{3}}=81.$

88. $\frac{1}{3}x^{\frac{3}{2}}=72.$

89. $x^{-\frac{1}{2}}=12.$

90. $\frac{1}{4}x^{\frac{2}{3}}=25.$

91. $2x^{-\frac{3}{2}}=\frac{1}{32}.$

92. $x^{-\frac{1}{2}}=6.$

93. $x^{-\frac{2}{3}}=144.$

94. $25x^{-\frac{2}{3}}=1.$

95. $x^{\frac{5}{2}}=243.$

96. $x^{\frac{5}{3}}+32=0.$

97. $x^{\frac{3}{4}}-a^6=0.$

98. $x^{\frac{6}{5}}-64=0.$

99. $x^{-\frac{3}{2}}-27=0.$

nplify, expressing results with positive exponents:

). $\left(\dfrac{\sqrt{3^{\frac{1}{3}}}}{\sqrt[3]{2^{-2}}}\right)^{-6}$.   102. $\left(\dfrac{16\,m^{-\frac{2}{3}}}{r^{-4}}\right)^{-\frac{3}{4}}$.   104. $\sqrt[3]{\dfrac{9^r \times 3^{2+r}}{27^r}}$.

$\left(\dfrac{9^{-8}}{x^{-4}y^{-2}}\right)^{-\frac{1}{2}}$.   103. $\left(\dfrac{m^{-2}n^{-\frac{2}{3}}}{x^{\frac{1}{6}}}\right)^{-3}$.   105. $\dfrac{3\,a^{\frac{1}{4}} \times 4\,a^{-1}}{6\sqrt[4]{a^5}}$.

$\dfrac{\sqrt[3]{a^2} \times \sqrt{b^3}}{\sqrt[4]{b^6} \times \sqrt[6]{a^{-2}}}$.   108. $\dfrac{x^2y^{\frac{3}{5}} \times \sqrt[3]{x^{-5}} \times x^{\frac{2}{3}}}{\sqrt[5]{y^{-2}} \times xy}$.

$\dfrac{(9^n \times 3^n \times 9) - 27^{n+1}}{3^2 \times 3^{3n}}$.   109. $\dfrac{9^{r+1}}{(3^{r-1})^{r+1}} \div \dfrac{3^{r+1}}{(3^r)^{r-1}}$.

110. $\dfrac{m^{\frac{1}{2}} - n^{\frac{1}{2}}}{\sqrt{m^3} - \sqrt{n^3}} \div \dfrac{\sqrt{m^3} + \sqrt{n^3}}{m^{\frac{1}{2}} + n^{\frac{1}{2}}}$.

111. $\left(2 + \dfrac{6}{\sqrt{x}}\right)^{-1} \div \dfrac{x - 6\sqrt{x}}{x - 3\,x^{\frac{1}{2}} - 18}$.

112. $\left(\dfrac{2\,a^{\frac{1}{3}} - b^{\frac{1}{3}}}{8\,a - b}\right)^{-1} - \left(\dfrac{1}{4\,a^{\frac{2}{3}}b^{\frac{2}{3}}}\right)^{-\frac{1}{2}}$.

113. $\left(\dfrac{c^{-3}x}{a^{-\frac{2}{3}}}\right)^{-\frac{3}{2}} \div \left(\dfrac{\sqrt[6]{a^3} \times \sqrt{c^{-\frac{1}{2}}}}{c^2x^{-1}}\right)^{-2}$.

114. $\sqrt{\left\{\left(\dfrac{r^{\frac{1}{3}}}{s^{\frac{1}{3}}}\right)^2 \div \dfrac{s^{-\frac{1}{2}}}{r^{-\frac{1}{3}}} \times \dfrac{\sqrt[4]{s^{-1}}}{\sqrt[3]{r}}\right\}^6}$.

115. $\left[\sqrt[4]{\dfrac{r^2n^{-\frac{3}{2}}}{l^{-1}\sqrt{m^5}}} \times \sqrt[3]{\dfrac{m^{-\frac{1}{3}}\sqrt{nr^3}}{m^{-\frac{13}{8}}l^{\frac{3}{4}}\sqrt[8]{n^7}}}\right]^{-4}$.

116. $\left[\sqrt{\left(\dfrac{nr^{-3}}{\sqrt[3]{m^{-2}}}\right)^{-3}} \div \left(\sqrt[4]{\dfrac{r^{-1}}{n^7}} \times \dfrac{\sqrt[6]{m^3}}{r^2}\right)^{-2}\right]^{-\frac{3}{5}}$.

Find the value of

$$\{(-2)^{-\frac{1}{4}} \div (-32)^{-\frac{1}{4}}\}^{-3} \div \{16^{\frac{5}{8}} \times (-8)^{\frac{5}{3}}\}^{-1}.$$

Extract the square root of

$$\frac{4\,x^2}{y} + \frac{\sqrt{x^3}}{y^{-\frac{1}{2}}} - 2\,x + \frac{1}{4\,y^{-1}} + x^3 - 4\sqrt{x^5y^{-1}}.$$

RADICALS

315. An indicated root of a number is called a **radical**; the number whose root is required is called the **radicand.**

$\sqrt{5a}$, $(x^6)^{\frac{1}{3}}$, $\sqrt[3]{a^2+2}$, and $(x+y)^{\frac{1}{4}}$ are radicals whose radicands are, respectively, $5a$, x^5, a^2+2, and $x+y$.

316. An expression that involves a radical, in any way, is called a **radical expression.**

317. In the discussion and treatment of radicals only *principal roots* will be considered.

Thus, $\sqrt{16}$ will be taken to represent only the principal square root of 16, or 4. The other square root will be denoted by $-\sqrt{16}$.

318. A number that is, or may be, expressed exactly without a root sign of any kind is called a **rational number**.

3, $\frac{1}{2}$, $a-b$, and $\sqrt{25}$ are rational numbers.

319. A number that cannot be expressed exactly without a root sign of some kind is called an **irrational number**.

$x^{\frac{1}{2}}$, $\sqrt[3]{\frac{1}{2}}$, $1+\sqrt{3}$, and $\sqrt{1+\sqrt{3}}$ are irrational numbers.

320. An expression is **irrational,** if it contains an irrational number, otherwise it is **rational.**

321. When the indicated root of a *rational* number cannot be exactly obtained, the expression is called a **surd.**

$\sqrt{2}$ is a surd, since 2 is rational but has no rational square root.

$\sqrt{1+\sqrt{3}}$ is not a surd, because $1+\sqrt{3}$ is not rational.

Radicals may be either rational or irrational, but surds are *always irrational.*

Both $\sqrt{4}$ *and* $\sqrt{3}$ are radicals but only $\sqrt{3}$ is a surd.

}. The **order** of a radical or of a surd is indicated by ndex of the root or by the denominator of the frac- exponent.

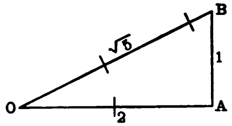
$\overline{+x}$ and $(b+x)^{\frac{1}{2}}$ are radicals of the second order.

. A surd of the second order is called a **quadratic surd**; e third order, a **cubic surd**; and of the fourth order, **adratic surd.**

. **Graphical representation of a quadratic surd.**

geometry it is shown that the hypotenuse of a right le is equal to the *square root of the sum of the squares of ıer two sides;* consequently, a quadratic surd may be rep- ;ed graphically by the *hypotenuse* of a right triangle whose two sides are such that the sum of their squares is equal radicand.

ıs, to represent $\sqrt{5}$ graphically, since it may be observed $5=2^2+1^2$, draw OA 2 units in ı, then draw AB 1 unit in length direction perpendicular to OA. OB, completing the right-angled gle OAB.



en the length of OB represents $\sqrt{5}$ in its relation to the length.

will be observed that $\sqrt{5}$ can be represented graphically by a line of length, though it cannot be represented exactly by decimal figures, $5=2.236\cdots$, an endless decimal.

EXERCISES

. Represent graphically:

$\sqrt{2}$. 3. $\sqrt{13}$. 5. $\sqrt{34}$. 7. $\sqrt{\frac{5}{4}}$.

$\sqrt{10}$. 4. $\sqrt{17}$. 6. $\sqrt{53}$. 8. $\sqrt{\frac{13}{36}}$.

. In the following pages it will be assumed that irrational numbers ıe same laws as rational numbers. For proofs of the generality e *laws, the reader is* referred to the author's *Advanced Algebra*.

327. A surd may contain a *rational factor*, that is, a factor whose radicand is a perfect power of the same degree as the radical. The rational factor may be removed and written as the coefficient of the irrational factor.

In $\sqrt{8} = \sqrt{4 \times 2}$ and $\sqrt[3]{54} = \sqrt[3]{27 \times 2}$, the rational factors are $\sqrt{4}$ and $\sqrt[3]{27}$, respectively; that is, $\sqrt{8} = 2\sqrt{2}$ and $\sqrt[3]{54} = 3\sqrt[3]{2}$.

328. A surd that has a rational coefficient is called a **mixed surd.**

$2\sqrt{2}$, $a\sqrt[3]{x^2}$, and $(a - b)\sqrt{a + b}$ are mixed surds.

329. A surd that has no rational coefficient except unity is called an **entire surd.**

$\sqrt{5}$, $\sqrt[3]{11}$, and $\sqrt{a^2 + x^2}$ are entire surds.

330. A radical is in its **simplest form** when the index of the root is as small as possible, and when the radicand is integral and contains no factor that is a perfect power whose exponent corresponds with the index of the root.

$\sqrt{7}$ is in its simplest form; but $\sqrt{\frac{3}{4}}$ is not in its simplest form, because $\frac{3}{4}$ is not integral in form; $\sqrt{8}$ is not in its simplest form, because the square root of 4, a factor of 8, may be extracted; $\sqrt[6]{25}$, or $25^{\frac{1}{6}}$, is not in its simplest form, because $25^{\frac{1}{6}} = (5^2)^{\frac{1}{6}} = 5^{\frac{2}{6}} = 5^{\frac{1}{3}}$, or $\sqrt[3]{5}$.

REDUCTION OF RADICALS

331. To reduce a radical to its simplest form.

EXERCISES

1. Reduce $\sqrt{20\,a^6}$ to its simplest form.

PROCESS

$$\sqrt{20\,a^6} = \sqrt{4\,a^6 \times 5} = \sqrt{4\,a^6} \times \sqrt{5} = 2\,a^3\sqrt{5}$$

Explanation. — Since the highest factor of $20\,a^6$ that is a perfect square is $4\,a^6$, $\sqrt{20\,a^6}$ is separated into two factors, a rational factor $\sqrt{4\,a^6}$, and an irrational factor $\sqrt{5}$, that is, § 291, $\sqrt{20\,a^6} = \sqrt{4\,a^6} \times \sqrt{5}$. *On extracting the square root of* $4\,a^6$ and *prefixing the root to the irrational factor as a coefficient,* the result is $2\,a^3\sqrt{5}$.

2. Reduce $\sqrt[3]{-864}$ to its simplest form.

PROCESS

$$\sqrt[3]{-864}=\sqrt[3]{-216\times 4}=\sqrt[3]{-216}\times\sqrt[3]{4}=-6\sqrt[3]{4}$$

RULE. — *Separate the radical into two factors one of which is its highest rational factor. Extract the required root of the rational factor, multiply the result by the coefficient, if any, of the given radical, and place the product as the coefficient of the irrational factor.*

Simplify:

3. $\sqrt{12}$.
4. $\sqrt{75}$.
5. $\sqrt[3]{16}$.
6. $\sqrt{128}$.
7. $\sqrt[3]{250}$.
8. $\sqrt[4]{32}$.
9. $\sqrt{162}$.
10. $\sqrt{18a^2}$.
11. $\sqrt{25b}$.
12. $\sqrt{98c^3}$.
13. $\sqrt{50a}$.
14. $\sqrt[5]{640}$.
15. $\sqrt{243a^5x^{10}}$.
16. $\sqrt[3]{128a^6b^4}$.
17. $(245a^6y^{-4})^{\frac{1}{2}}$.
18. $(a^3+5a^2)^{\frac{1}{2}}$.
19. $\sqrt{18x-9}$.
20. $\sqrt[3]{x^6-2x^3}$.
21. $\sqrt{5x^2-10xy+5y^2}$.
22. $(3am^2+6am+3a)^{\frac{1}{2}}$.

23. Reduce $\sqrt{\dfrac{a^2}{2y^3}}$ to its simplest form.

PROCESS

$$\sqrt{\frac{a^2}{2y^3}}=\sqrt{\frac{a^2\times 2y}{2y^3\times 2y}}=\sqrt{\frac{a^2}{4y^4}}\times\sqrt{2y}=\frac{a}{2y^2}\sqrt{2y}$$

EXPLANATION. — Since a radical is not in its simplest form when the expression under the radical sign is fractional, the denominator must be removed; and since the radical is of the second degree, the denominator must be made a perfect square. The smallest factor that will accomplish this is $2y$. On multiplying the terms of the fraction by this factor, the largest rational factor of the resulting radical is found to be $\sqrt{\dfrac{a^2}{4y^4}}$, which is equal to $\dfrac{a}{2y^2}$. Therefore, the irrational factor is $\sqrt{2y}$, and its coefficient is $\dfrac{a}{2y^2}$.

Simplify:

24. $\sqrt{\frac{1}{2}}$.

25. $\sqrt{\frac{1}{5}}$.

26. $\sqrt{\frac{2}{3}}$.

27. $\sqrt{\frac{3}{8}}$.

28. $\sqrt[3]{\frac{5}{12}}$.

29. $\sqrt[3]{\frac{4}{5}}$.

30. $\sqrt{\frac{2a^3}{b}}$.

31. $\sqrt{\frac{5x^4y^2}{2a^2}}$.

32. $\sqrt[4]{\frac{x}{y}}$.

33. $\sqrt{\frac{2}{3y^5}}$.

34. $\sqrt{\frac{4a}{3x^2}}$.

35. $\sqrt{\frac{3x}{50a^3y}}$.

36. $(a+b)\sqrt{\frac{a+b}{a-b}}$.

37. $\frac{(a+b)^2}{a-b}\sqrt[3]{\frac{a+b}{(a-b)^2}}$.

332. Although $\frac{3}{6}=\frac{1}{2}$, it does not follow without proof that $64^{\frac{3}{6}}=64^{\frac{1}{2}}$, for each fractional exponent denotes a power of a root of 64, and the roots and powers taken are not the same for $64^{\frac{1}{2}}$ as for $64^{\frac{3}{6}}$. By trial, however, it is found that each number is equal to 8; and in general it may be proved that

$$a^{\frac{pm}{qm}}=a^{\frac{p}{q}}; \text{ that is,}$$

A number having a fractional exponent is not changed in value by reducing the fractional exponent to higher or lower terms.

EXERCISES

333. 1. Reduce $\sqrt[6]{9a^2}$ to its simplest form.

PROCESS

$$\sqrt[6]{9a^2}=\sqrt[6]{(3a)^2}=(3a)^{\frac{2}{6}}=(3a)^{\frac{1}{3}}=\sqrt[3]{3a}$$

2. Reduce $\sqrt[9]{64a^6b^{15}}$ to its simplest form.

PROCESS

$$\sqrt[9]{64a^6b^{15}}=\sqrt[9]{2^6a^6b^6b^9}=b(2ab)^{\frac{6}{9}}=b(2ab)^{\frac{2}{3}}=b\sqrt[3]{4a^2b^2}$$

Simplify:

3. $\sqrt[4]{36}$.

4. $\sqrt[4]{25}$.

5. $\sqrt[4]{1600}$.

6. $\sqrt[6]{27a^3}$.

7. $\sqrt[4]{9a^2b^2c^6}$.

8. $\sqrt[4]{121a^6x^4}$.

334. Simplify:

1. $\sqrt{600}$.
2. $\sqrt{500}$.
3. $\sqrt[5]{160}$.
4. $\sqrt[3]{3000}$.
5. $\sqrt[3]{189}$.
6. $\sqrt{84}$.
7. $\sqrt[3]{72}$.
8. $\sqrt[3]{192}$.
9. $\sqrt[4]{144}$.
10. $\sqrt[6]{81}$.
11. $\sqrt[6]{343}$.
12. $\sqrt[4]{289}$.
13. $\sqrt{\frac{1}{8}}$.
14. $\sqrt{\frac{1}{x^3}}$.
15. $\sqrt[3]{\frac{a}{3b^2}}$.
16. $\sqrt{405a^5y^2}$.
17. $(135x^4y^5)^{\frac{1}{3}}$.
18. $\sqrt{8-20b^2}$.
19. $5\sqrt{4a^2+4}$.
20. $\sqrt[6]{a^4b^2c^4d^8}$.
21. $(16x-16)^{\frac{1}{2}}$.
22. $\dfrac{2y}{x-2y}\sqrt{\dfrac{x-2y}{2y}}$.
23. $\sqrt{27c^2-36c+12}$.
24. $\sqrt[4]{x^2-2xy+y^2}$.
25. $(1-x^3)\sqrt{\dfrac{1-x+x^2}{1+x+x^2}}$.
26. $(4a^3-24a^2x+36ax^2)^{\frac{1}{2}}$.
27. $(x^4y-3x^3y^2+3x^2y^3-xy^4)^{\frac{1}{3}}$.

335. To reduce a mixed surd to an entire surd.

EXERCISES

1. Express $2a\sqrt{5b}$ as an entire surd.

PROCESS

$$2a\sqrt{5b}=\sqrt{4a^2}\sqrt{5b}=\sqrt{4a^2\times 5b}=\sqrt{20a^2b}$$

RULE. — *Raise the coefficient to a power corresponding to the index of the given radical, and introduce the result under the radical sign as a factor.*

Express as entire surds:

2. $2\sqrt{2}$.
3. $3\sqrt{5}$.
4. $5\sqrt{2}$.
5. $3\sqrt[4]{2}$.
6. $3\sqrt[3]{3}$.
7. $4\sqrt{5}$.
8. $\frac{1}{2}\sqrt{8}$.
9. $a^2\sqrt[3]{b}$.
10. $\frac{1}{2}\sqrt{2}$.
11. $\frac{3}{4}\sqrt{x^5}$.
12. $\frac{1}{2}\sqrt{bc}$.
13. $\frac{3}{4}\sqrt{\frac{7}{9}}$.
14. $\frac{4}{5}\sqrt{4\frac{3}{8}}$.
15. $\frac{3}{2}\sqrt{\frac{33}{27}a^2}$.
16. $\frac{2}{3}\sqrt[3]{1\frac{1}{8}}$.
17. $\frac{2}{3}\sqrt[4]{3\frac{3}{8}}$.
18. $\dfrac{x+y}{x-y}\sqrt{\dfrac{x-y}{x+y}}$.
19. $\dfrac{a+4}{a-4}\sqrt{1-\dfrac{8}{a+4}}$.
20. $\dfrac{1}{ab}(a-b)^{\frac{1}{2}}$.

336. To reduce radicals to the same order.

EXERCISES

1. Reduce $\sqrt[4]{3}$, $\sqrt{2}$, and $\sqrt[3]{4}$ to radicals of the same order.

PROCESS

$$\sqrt[4]{3}=3^{\frac{1}{4}}=3^{\frac{3}{12}}=\sqrt[12]{3^3}=\sqrt[12]{27}$$

$$\sqrt{2}=2^{\frac{1}{2}}=2^{\frac{6}{12}}=\sqrt[12]{2^6}=\sqrt[12]{64}$$

$$\sqrt[3]{4}=4^{\frac{1}{3}}=4^{\frac{4}{12}}=\sqrt[12]{4^4}=\sqrt[12]{256}$$

RULE. — *Express the given radicals with fractional exponents having a common denominator.*

Raise each number to the power indicated by the numerator of its fractional exponent, and indicate the root expressed by the common denominator.

Reduce to radicals of the same order:

2. $\sqrt{2}$ and $\sqrt[4]{3}$.
3. $\sqrt{5}$ and $\sqrt[3]{6}$.
4. $\sqrt[4]{7}$ and $\sqrt{10}$.
5. $\sqrt[6]{10}$, $\sqrt{2}$, and $\sqrt[3]{5}$.
6. $\sqrt[6]{4}$, $\sqrt[4]{2}$, and $\sqrt{3}$.
7. $\sqrt[10]{13}$, $\sqrt{5}$, and $\sqrt[5]{4}$.
8. $\sqrt{3}$, $\sqrt[3]{5}$, and $\sqrt[4]{\frac{1}{27}}$.
9. $\sqrt{ab}$, $\sqrt[3]{ab^2}$, and $\sqrt[4]{2}$.
10. $\sqrt{a}$, $\sqrt[3]{b}$, $\sqrt[4]{x}$, and $\sqrt[6]{y}$.
11. $\sqrt[3]{a+b}$ and $\sqrt{x+y}$.
12. $\sqrt{\frac{2}{3}}$, $\sqrt[3]{\frac{1}{16}x}$, and $2\sqrt{5}$.
13. $\sqrt[n]{x}$, $\sqrt{xy}$, and $\sqrt[n]{x^2y^2}$.
14. $(a+b)\sqrt{a-b}$, and $\sqrt[3]{a-b}$.
15. $\sqrt{a+b}$, $\sqrt[4]{a^2+b^2}$, and $\sqrt{a-b}$.
16. Which is greater, $\sqrt[5]{5}$ or $\sqrt{2}$? $\sqrt[3]{4}$ or $\sqrt{3}$?
17. Which is greater, $\sqrt[3]{3}$ or $\sqrt[4]{4}$? $3\sqrt{2}$ or $2\sqrt[3]{4}$?

Arrange in order of value:

18. $\sqrt[3]{3}$, $\sqrt{2}$, and $\sqrt[6]{7}$.
19. $\sqrt{2}$, $\sqrt[3]{4}$, and $\sqrt[4]{5}$.
20. $\sqrt[3]{2}$, $\sqrt[5]{3}$, and $\sqrt[15]{30}$.
21. $\sqrt{2}$, $\sqrt[3]{5}$, $\sqrt{2\frac{1}{2}}$, and $\sqrt[3]{4}$.
22. $\sqrt{7}$, $\sqrt[4]{48}$, $\sqrt[3]{4}$, and $\sqrt[6]{63}$.
23. $\sqrt[3]{4}$, $\sqrt{2}$, $\sqrt[4]{5}$, $\sqrt[6]{13}$, and $\sqrt[12]{150}$.

ADDITION AND SUBTRACTION OF RADICALS

337. Radicals that in their simplest form are of the same order and have the same radicand are called **similar radicals.**

Thus, $2\sqrt{3}$, $a\sqrt{3}$, and $7\sqrt{3}$ are similar radicals.

338. PRINCIPLE. — *Only similar radicals can be united into one term by addition or subtraction.*

EXERCISES

339. 1. Find the sum of $\sqrt{50}$, $2\sqrt[6]{8}$, and $6\sqrt{\frac{1}{2}}$.

PROCESS

$$\begin{aligned} \sqrt{50} &= 5\sqrt{2} \\ 2\sqrt[6]{8} &= 2\sqrt{2} \\ 6\sqrt{\tfrac{1}{2}} &= 3\sqrt{2} \\ \hline \text{Sum} &= 10\sqrt{2} \end{aligned}$$

EXPLANATION. — To ascertain whether the given expressions are similar radicals, each may be reduced to its simplest form. Since, in their simplest form, they are of the same degree and have the same radicand, they are similar, and their sum is obtained by prefixing the sum of the coefficients to the common radical factor.

Find the sum of:

2. $\sqrt{50}$, $\sqrt{18}$, and $\sqrt{98}$.

3. $\sqrt{27}$, $\sqrt{12}$, and $\sqrt{75}$.

4. $\sqrt{20}$, $\sqrt{80}$, and $\sqrt{45}$.

5. $\sqrt{28}$, $\sqrt{63}$, and $\sqrt{700}$.

6. $\sqrt[3]{250}$, $\sqrt[3]{16}$, and $\sqrt[3]{54}$.

7. $\sqrt[3]{128}$, $\sqrt[3]{686}$, and $\sqrt[3]{\frac{1}{4}}$.

8. $\sqrt[3]{135}$, $\sqrt[3]{320}$, and $\sqrt[3]{625}$.

9. $\sqrt[3]{500}$, $\sqrt[3]{108}$, and $\sqrt[3]{-32}$.

10. $\sqrt{\frac{1}{2}}$, $\sqrt{12\frac{1}{2}}$, $\sqrt{\frac{1}{8}}$, and $\sqrt{1\frac{1}{8}}$.

11. $\sqrt{\frac{1}{3}}$, $\sqrt{75}$, $\frac{2}{3}\sqrt{3}$, and $\sqrt{12}$.

12. $\sqrt{\frac{3}{4}}$, $\frac{1}{3}\sqrt{3}$, $\frac{7}{6}\sqrt[4]{9}$, and $\sqrt{147}$.

13. $\sqrt[3]{40}$, $\sqrt{28}$, $\sqrt[6]{25}$, and $\sqrt{175}$.

14. $\sqrt[3]{375}$, $\sqrt{44}$, $\sqrt[3]{192}$, and $\sqrt{99}$.

Simplify:

15. $\sqrt{245}-\sqrt{405}+\sqrt{45}$.

16. $\sqrt{12}+3\sqrt{75}-2\sqrt{27}$.

17. $5\sqrt{72}+3\sqrt{18}-\sqrt{50}$.

18. $\sqrt[3]{128}+\sqrt[3]{686}-\sqrt[3]{54}$.

19. $\sqrt{112}-\sqrt{343}+\sqrt{448}$.

20. $\sqrt[3]{135}-\sqrt[3]{625}+\sqrt[3]{320}$.

21. $\sqrt[3]{\frac{8}{5}}+\sqrt[3]{\frac{1}{5}}+\sqrt[3]{5\frac{2}{5}}$.

22. $\sqrt[3]{864}-\sqrt[3]{4000}+\sqrt[3]{32}$.

23. $\sqrt[3]{128\,x}+\sqrt[3]{375\,x}-\sqrt[3]{54\,x}$.

24. $\sqrt{\frac{a}{x^2}}+\sqrt{\frac{a}{y^2}}-\sqrt{\frac{a}{z^2}}$.

25. $\sqrt{\frac{ax^4}{by^2}}-\sqrt{\frac{16\,ax^2}{by^2}}+\sqrt{\frac{4\,ax^2}{by^2}}$.

26. $\sqrt{\frac{a}{bc}}+\sqrt{\frac{b}{ac}}+\sqrt{\frac{c}{ab}}$.

27. $\sqrt{(a+b)^2c}-\sqrt{(a-b)^2c}$.

28. $6\sqrt[3]{\frac{40}{27}}+4\sqrt[3]{\frac{10}{16}}-8\sqrt[3]{\frac{675}{320}}$.

29. $\sqrt[5]{-96\,x^4}+2\sqrt[5]{3\,x^4}-\sqrt[3]{5\,x}+\sqrt[3]{40\,x^4}$.

30. $\sqrt[3]{abx}-\sqrt[6]{a^2b^2x^2}+\sqrt[9]{8\,a^3b^3x^3}$.

31. $\sqrt{3\,x^3+30\,x^2+75\,x}-\sqrt{3\,x^3-6\,x^2+3\,x}$.

32. $\sqrt{5\,a^5+30\,a^4+45\,a^3}-\sqrt{5\,a^5-40\,a^4+80\,a^3}$.

33. $\sqrt{50}+\sqrt[6]{9}-4\sqrt{\frac{1}{2}}+\sqrt[3]{-24}+\sqrt[9]{27}-\sqrt[4]{64}$.

34. $\sqrt{\frac{2}{3}}+6\sqrt{\frac{5}{4}}-\frac{1}{5}\sqrt{18}+\sqrt[4]{36}-\sqrt[8]{\frac{16}{81}}+\sqrt[6]{125}-2\sqrt{\frac{2}{25}}$.

35. $(\frac{8}{3})^{\frac{1}{2}}-(\frac{2}{3})^{-\frac{1}{2}}+\sqrt{(\frac{8}{27})^{-1}}+\sqrt{1.35}-\sqrt[4]{(1\frac{2}{3})^{-2}}$.

36. $5\cdot 2^{-\frac{2}{3}}+2^{-\frac{5}{3}}+3\cdot 2^{-\frac{8}{3}}+3\cdot 5^{-1}\cdot 2^{\frac{1}{5}}+\sqrt[5]{\frac{64}{3125}}$.

MULTIPLICATION OF RADICALS

340. $$a^{\frac{1}{2}}\times a^{\frac{1}{3}}=a^{\frac{1}{2}+\frac{1}{3}}=a^{\frac{3}{6}+\frac{2}{6}}=a^{\frac{5}{6}}.$$

That is, $$\sqrt{a}\times\sqrt[3]{a}=\sqrt[6]{a^3}\times\sqrt[6]{a^2}=\sqrt[6]{a^5}.$$

Since fractional exponents to be united by addition must be *expressed* with a common denominator, radicals to be united by *multiplication* must be expressed with a common root index.

EXERCISES

341. PROCESSES. — 1. $\sqrt{7} \times \sqrt{5} = \sqrt{35}$.

2. $5\sqrt{3} \times 2\sqrt{15} = 10\sqrt{45} = 10 \times 3\sqrt{5} = 30\sqrt{5}$.

3. $2\sqrt{3} \times 3\sqrt[3]{2} = 2\sqrt[6]{27} \times 3\sqrt[6]{4} = 6\sqrt[6]{108}$.

RULE. — *If the radicals are not of the same order, reduce them to the same order.*

Multiply *the coefficients for the coefficient of the product and the* ***radicands*** *for the radical factor of the product; simplify the result, if necessary.*

Multiply:

4. $\sqrt{2}$ by $\sqrt{8}$.
5. $\sqrt{2}$ by $\sqrt{6}$.
6. $\sqrt{3}$ by $\sqrt{15}$.
7. $2\sqrt{5}$ by $3\sqrt{10}$.
8. $3\sqrt{20}$ by $2\sqrt{2}$.
9. $\sqrt{2}$ by $3\sqrt[3]{3}$.
10. $2\sqrt[3]{3}$ by $3\sqrt[3]{45}$.
11. $2\sqrt[4]{6}$ by $3\sqrt{6}$.
12. $3\sqrt{3}$ by $2\sqrt[3]{5}$.
13. $\sqrt[4]{5}$ by $\sqrt[6]{10}$.
14. $2\sqrt[3]{250}$ by $\sqrt{2}$.
15. $2\sqrt[3]{24}$ by $\sqrt[3]{18}$.
16. $2\sqrt[5]{2}$ by $\sqrt[10]{512}$.
17. $\sqrt{2xy}$ by $3\sqrt[3]{x^2y^3}$.

Find the value of:

18. $\sqrt{mn} \times \sqrt[4]{m^2n} \times \sqrt[8]{mn^4}$.
19. $\sqrt{2axy} \times \sqrt[3]{xy} \times \sqrt[4]{a^2xy}$.
20. $\sqrt{x^{-1}y} \times \sqrt[3]{x^{-2}y^2} \times \sqrt{x^{-3}y^3}$.
21. $\sqrt{a-b} \times \sqrt[4]{a^2b^2} \times \sqrt[4]{(a-b)^{-2}}$.

22. $\sqrt{\frac{2}{3}} \times \sqrt{\frac{4}{5}} \times \sqrt{\frac{3}{4}}$.
23. $\sqrt{\frac{1}{2}} \times \sqrt{\frac{4}{5}} \times \sqrt{\frac{7}{2}}$.
24. $\sqrt[3]{\frac{2}{3}} \times \sqrt[3]{\frac{3}{4}} \times \sqrt{\frac{1}{2}}$.
25. $\sqrt[3]{\frac{1}{3}} \times \sqrt[6]{\frac{3}{2}} \times \sqrt{\frac{2}{3}}$.
26. $16^{\frac{1}{3}} \times 2^{\frac{1}{2}} \times 32^{\frac{5}{6}}$.
27. $27^{\frac{1}{4}} \times 9^{\frac{1}{3}} \times 81^{\frac{1}{3}}$.

28. Multiply $2\sqrt{2}+3\sqrt{3}$ by $5\sqrt{2}-2\sqrt{3}$.

SOLUTION

$$\begin{array}{lll}
2\sqrt{2}+3\sqrt{3} & & \\
\underline{5\sqrt{2}-2\sqrt{3}} & & \\
20 & +15\sqrt{6} & \\
 & \underline{-\ \ 4\sqrt{6}-18} & \\
20 & +11\sqrt{6}-18=2+11\sqrt{6}. &
\end{array}$$

Multiply:

29. $\sqrt{5}+\sqrt{3}$ by $\sqrt{5}-\sqrt{3}$.

30. $\sqrt{7}+\sqrt{2}$ by $\sqrt{7}-\sqrt{2}$.

31. $\sqrt{6}-\sqrt{5}$ by $\sqrt{6}-\sqrt{5}$.

32. $5-\sqrt{5}$ by $1+\sqrt{5}$.

33. $4\sqrt{7}+1$ by $4\sqrt{7}-1$.

34. $2\sqrt{2}+\sqrt{3}$ by $4\sqrt{2}+\sqrt{3}$.

35. $2\sqrt{3}+3\sqrt{5}$ by $3\sqrt{3}+2\sqrt{5}$.

36. $3a+\sqrt{5}$ by $2a-\sqrt{5}$.

37. $2\sqrt{6}-3\sqrt{5}$ by $4\sqrt{3}-\sqrt{10}$.

38. $a^2-ab\sqrt{2}+b^2$ by $a^2+ab\sqrt{2}+b^2$.

39. $x-\sqrt{xyz}+yz$ by $\sqrt{x}+\sqrt{yz}$.

40. $x\sqrt{x}-x\sqrt{y}+y\sqrt{x}-y\sqrt{y}$ by $\sqrt{x}+\sqrt{y}$.

Expand:

41. $(\sqrt{3+\sqrt{5}})(\sqrt{3-\sqrt{5}})$.

43. $(\sqrt{6+\sqrt{11}})(\sqrt{6-\sqrt{1}}$

42. $(\sqrt{9+\sqrt{6}})(\sqrt{9-\sqrt{6}})$.

44. $(\sqrt{5a+a\sqrt{5}})(\sqrt{5a-a}$

45. $(\sqrt{7c+\sqrt{5c^2}})(\sqrt{7c-\sqrt{5c^2}})$.

46. $(\sqrt{14x+x\sqrt{27}})(\sqrt{14x-x\sqrt{27}})$.

DIVISION OF RADICALS

342. $$a^{\frac{1}{2}} \div a^{\frac{1}{3}} = a^{\frac{1}{2}-\frac{1}{3}} = a^{\frac{3}{6}-\frac{2}{6}} = a^{\frac{1}{6}}.$$

That is, $$\sqrt{a} \div \sqrt[3]{a} = \sqrt[6]{a^3} \div \sqrt[6]{a^2} = \sqrt[6]{a^3 \div a^2} = \sqrt[6]{a}.$$

In division, when one fractional exponent is subtracted from another, the exponents must be expressed with a common denominator. When one radical is divided by another, the radicals must be expressed with a common root index.

EXERCISES

343. Processes. — 1. $\sqrt{60} \div \sqrt{12} = \sqrt{5}$.

2. $\sqrt[3]{2} \div \sqrt{2} = \sqrt[6]{4} \div \sqrt[6]{8} = \sqrt[6]{\frac{4}{8}} = \sqrt[6]{\frac{32}{64}} = \frac{1}{2}\sqrt[6]{32}$.

3. $$\sqrt[3]{x^2} \div \sqrt[4]{y} = \frac{(x^2)^{\frac{1}{3}}}{(y)^{\frac{1}{4}}} = \frac{(x^8)^{\frac{1}{12}}}{(y^3)^{\frac{1}{12}}} = \left(\frac{x^8}{y^3}\right)^{\frac{1}{12}} = \sqrt[12]{\frac{x^8}{y^3}} = \frac{1}{y}\sqrt[12]{x^8y^9}.$$

Rule. — *If necessary, reduce the radicals to the same order.*

To the quotient of the coefficients annex the quotient of the radicands written under the common radical sign, and reduce the result to its simplest form.

Find quotients:

4. $\sqrt{50} \div \sqrt{8}$.
5. $\sqrt{72} \div 2\sqrt{6}$.
6. $4\sqrt{5} \div \sqrt{40}$.
7. $6\sqrt{7} \div \sqrt{126}$.
8. $\sqrt[3]{4} \div \sqrt{2}$.
9. $7\sqrt[3]{135} \div \sqrt[3]{9}$.
10. $7\sqrt{75} \div 5\sqrt{28}$.
11. $\sqrt[3]{16} \div \sqrt[6]{32}$.
12. $2\sqrt[3]{12} \div \sqrt{8}$.
13. $\sqrt[3]{ax} \div \sqrt{xy}$.
14. $\sqrt{2ab^3} \div \sqrt[4]{a^4b^4}$.
15. $\sqrt[3]{a^2x^2} \div \sqrt{2ax}$.
16. $\sqrt[3]{9a^2b^2} \div \sqrt{3ab}$.
17. $\sqrt[4]{4x^2y^2} \div \sqrt[3]{2xy}$.
18. $\sqrt{a-b} \div \sqrt{a+b}$.
19. $3\sqrt[3]{\frac{2}{3}} \div \sqrt{\frac{3}{4}}$.

20. Divide $\sqrt{15} - \sqrt{3}$ by $\sqrt{3}$.

21. Divide $\sqrt{6} - 2\sqrt{3} + 4$ by $\sqrt{2}$.

22. Divide $\sqrt{2} + 2 + \frac{1}{3}\sqrt{42}$ by $\frac{1}{3}\sqrt{6}$.

23. Divide $5\sqrt{2} + 5\sqrt{3}$ by $\sqrt{10} + \sqrt{15}$.

24. Divide $5 + 5\sqrt{30} + 36$ by $\sqrt{5} + 2\sqrt{6}$.

INVOLUTION AND EVOLUTION OF RADICALS

344. In finding powers and roots of radicals, it is frequent convenient to use fractional exponents.

EXERCISES

345. 1. Find the cube of $2\sqrt{ax^5}$.

SOLUTION. $(2\sqrt{ax^5})^3 = 2^3(a^{\frac{1}{2}}x^{\frac{5}{2}})^3 = 8\,a^{\frac{3}{2}}x^{\frac{15}{2}} = 8\sqrt{a^3x^{15}} = 8\,ax^7\sqrt{a}$

2. Find the square of $3\sqrt[6]{x^5}$.

SOLUTION. $(3\sqrt[6]{x^5})^2 = 9(x^{\frac{5}{6}})^2 = 9\,x^{\frac{5}{3}} = 9\sqrt[3]{x^5} = 9\,x\sqrt[3]{x^2}$.

3. Obtain by involution the cube of $\sqrt{2} + 1$.

SOLUTION

$$\begin{aligned}(\sqrt{2} + 1)^3 &= (\sqrt{2})^3 + 3(\sqrt{2})^2 \cdot 1 + 3\sqrt{2} \cdot 1^2 + 1^3 \\ &= 2\sqrt{2} + 6 + 3\sqrt{2} + 1 \\ &= 7 + 5\sqrt{2}.\end{aligned}$$

In such cases expand by the binomial formula.

Square:	Cube:	Involve as indicate
4. $3\sqrt{ab}$.	9. $2\sqrt{5}$.	14. $(-2\sqrt{2ab})^4$.
5. $2\sqrt[3]{3x}$.	10. $3\sqrt{2}$.	15. $(-\sqrt{2}\sqrt[6]{x})^3$.
6. $x\sqrt[3]{2x^3}$.	11. $2\sqrt[3]{a^2}$.	16. $(-\sqrt{2}\sqrt[3]{ax^2})^4$.
7. $n^2\sqrt{4b}$.	12. $\sqrt[4]{a^2b^3}$.	17. $(-2\sqrt{x}\sqrt[3]{y})^5$.
8. $a\sqrt[4]{a^2b}$.	13. $\sqrt[6]{4n^3}$.	18. $(-3a^{\frac{n}{2}}x^{\frac{n}{3}})^6$.

Expand:

19. $(2+\sqrt{6})^2$. 22. $(2-\sqrt{3})^3$. 25. $(\sqrt{x}\pm 1)^2$.

20. $(2+\sqrt{2})^2$. 23. $(\sqrt{7}-\sqrt{6})^2$. 26. $(\sqrt{a}-\sqrt{b})^3$.

21. $(2+\sqrt{5})^3$. 24. $(2\sqrt{2}-\sqrt{3})^2$. 27. $(\sqrt{x}\pm 1)^3$.

28. What is the fourth root of $\sqrt{2x}$?

SOLUTION. $\sqrt[4]{\sqrt{2x}}=[(2x)^{\frac{1}{2}}]^{\frac{1}{4}}=(2x)^{\frac{1}{8}}=\sqrt[8]{2x}.$

Find the square root of:

29. $\sqrt{2}$. 32. $\sqrt[3]{x^n}$.

30. $\sqrt[3]{5}$. 33. $\sqrt[5]{x^{12}}$.

31. $\sqrt[6]{x^2}$. 34. $\sqrt[n]{a^n x^2}$.

Find the cube root of:

35. $\sqrt{2x}$. 38. $-27\sqrt{x^6}$.

36. $\sqrt{7a^3}$. 39. $-\sqrt{a^n b^n}$.

37. $\sqrt[4]{8m^3x^3}$. 40. $-64\sqrt[5]{a^3y^3}$.

Simplify the following indicated roots:

41. $\sqrt{\sqrt[3]{4a^2x^4}}$. 43. $(\sqrt{8a^3x^3})^{\frac{1}{3}}$.

42. $\sqrt[3]{\sqrt{a^{12}x^4}}$. 44. $(\sqrt{x^m y^m})^{\frac{1}{mn}}$.

45. $\sqrt{\left(\dfrac{x^n b^2}{a^{-2}y^n}\right)^{\frac{2}{n}}}$.

Rationalization

346. Suppose that it is required to find the approximate value of $\dfrac{1}{\sqrt{3}}$, having given $\sqrt{3}=1.732\cdots$.

$$\begin{array}{r|l|l} 1.732\cdots & 1.000000 & .577\cdots \\ & 8660 & \\ & \cdots & \end{array} \qquad \begin{array}{r} 3)\underline{1.732\cdots} \\ .577\cdots \end{array}$$

We may obtain a decimal approximately equal to $\dfrac{1}{\sqrt{3}}$, as in the first process (incomplete), by dividing 1 by $1.732\cdots$; but a great saving of labor may be effected by first changing the fraction to an equivalent fraction having a *rational* denominator, thus:

$$\frac{1}{\sqrt{3}}=\frac{1\cdot\sqrt{3}}{\sqrt{3}\cdot\sqrt{3}}=\frac{\sqrt{3}}{3},$$

and employing the second process.

347. The process of multiplying a surd expression by any number that will make the product rational is called **rationalization.**

348. The factor by which a surd expression is multiplied to render the product rational is called the **rationalizing factor.**

349. The process of reducing a fraction having an irrational denominator to an equal fraction having a rational denominator is called **rationalizing the denominator.**

EXERCISES

350. Find the value of each of the following to the nearest fifth decimal place, taking $\sqrt{2} = 1.41421$, $\sqrt{3} = 1.73205$, and $\sqrt{5} = 2.23607$:

1. $\dfrac{5}{\sqrt{2}}$.
3. $\dfrac{6}{\sqrt{8}}$.
5. $\dfrac{15}{\sqrt{50}}$.
2. $\dfrac{2}{\sqrt{5}}$.
4. $\dfrac{10}{\sqrt{45}}$.
6. $\dfrac{1}{\sqrt{125}}$.

Rationalize the denominator of each of the following, using the smallest, or lowest, rationalizing factor possible:

7. $\dfrac{1}{\sqrt{x^7}}$.
9. $\dfrac{\sqrt[3]{6}}{\sqrt{12}}$.
11. $\dfrac{\sqrt{a+b}}{\sqrt{a-b}}$.
8. $\dfrac{ax}{\sqrt{2a^3x}}$.
10. $\dfrac{\sqrt{a}}{\sqrt[3]{ax^2}}$.
12. $\sqrt{1-\dfrac{4}{x+2}}$.

351. A binomial, one or both of whose terms are surds, is called a **binomial surd.**

$\sqrt{2} + \sqrt{5}$, $2 + \sqrt{5}$, $\sqrt[3]{2} + 1$, and $\sqrt{3} - \sqrt[3]{2}$ are binomial surds.

352. A binomial surd whose surd or surds are of the second order is called a **binomial quadratic surd.**

$\sqrt{2} + \sqrt{5}$ and $2 + \sqrt{5}$ are binomial quadratic surds.

353. Two binomial quadratic surds that differ only in the *sign of one* of the terms are called **conjugate surds.**

$3 + \sqrt{5}$ and $3 - \sqrt{5}$ are conjugate surds; also $\sqrt{3} + \sqrt{2}$ and $\sqrt{3} - \sqrt{2}$.

354. *The product of any two conjugate surds is rational.*

For, by § 114, $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b}) = (\sqrt{a})^2 - (\sqrt{b})^2 = a - b.$

Hence, *a binomial quadratic surd may be rationalized by multiplying it by its conjugate.*

EXERCISES

355. 1. Rationalize the denominator of $\dfrac{2}{3-\sqrt{5}}$.

SOLUTION

$$\frac{2}{3-\sqrt{5}} = \frac{2(3+\sqrt{5})}{(3-\sqrt{5})(3+\sqrt{5})} = \frac{2(3+\sqrt{5})}{9-5} = \frac{3+\sqrt{5}}{2}.$$

2. Rationalize the denominator of $\dfrac{\sqrt{7}-\sqrt{3}}{\sqrt{7}+\sqrt{3}}$.

SOLUTION

$$\frac{\sqrt{7}-\sqrt{3}}{\sqrt{7}+\sqrt{3}} = \frac{(\sqrt{7}-\sqrt{3})(\sqrt{7}-\sqrt{3})}{(\sqrt{7}+\sqrt{3})(\sqrt{7}-\sqrt{3})} = \frac{7-2\sqrt{21}+3}{7-3} = \frac{5-\sqrt{21}}{2}.$$

Rationalize the denominator of:

3. $\dfrac{3}{2+\sqrt{3}}$.

4. $\dfrac{5}{\sqrt{5}-\sqrt{3}}$.

5. $\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$.

6. $\dfrac{5-3\sqrt{2}}{2-\sqrt{2}}$.

7. $\dfrac{a-2\sqrt{b}}{a+2\sqrt{b}}$.

8. $\dfrac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}$.

9. $\dfrac{4\sqrt{2}+6\sqrt{3}}{3\sqrt{3}-2\sqrt{2}}$.

10. $\dfrac{x-\sqrt{x^2-1}}{x+\sqrt{x^2-1}}$.

11. $\dfrac{\sqrt{a^2+a+1}-1}{\sqrt{a^2+a+1}+1}$.

12. $\dfrac{\sqrt{x+y}-\sqrt{x-y}}{\sqrt{x+y}+\sqrt{x-y}}$.

Reduce to a decimal, to the nearest thousandth:

13. $\dfrac{8-\sqrt{3}}{2-\sqrt{3}}$.

14. $\dfrac{4}{3+\sqrt{5}}$.

15. $\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$.

16. Rationalize the denominator of $\dfrac{\sqrt{2}-\sqrt{3}-\sqrt{5}}{\sqrt{2}+\sqrt{3}+\sqrt{5}}$.

SOLUTION

$$\frac{\sqrt{2}-\sqrt{3}-\sqrt{5}}{\sqrt{2}+\sqrt{3}+\sqrt{5}}=\frac{(\sqrt{2}-\sqrt{5})-\sqrt{3}}{(\sqrt{2}+\sqrt{3})+\sqrt{5}}\times\frac{(\sqrt{2}-\sqrt{5})+\sqrt{3}}{(\sqrt{2}+\sqrt{3})-\sqrt{5}}$$

$$=\frac{2-2\sqrt{10}+5-3}{2+2\sqrt{6}+3-5}=\frac{4-2\sqrt{10}}{2\sqrt{6}}$$

$$=\frac{2-\sqrt{10}}{\sqrt{6}}=\frac{2\sqrt{6}-2\sqrt{15}}{6}=\frac{\sqrt{6}-\sqrt{15}}{3}.$$

Rationalize the denominator of:

17. $\dfrac{\sqrt{2}-\sqrt{5}-\sqrt{7}}{\sqrt{2}+\sqrt{5}+\sqrt{7}}$.

18. $\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}-\sqrt{6}}$.

19. $\dfrac{1}{\sqrt{2}+\sqrt{3}+\sqrt{5}}$.

20. $\dfrac{2\sqrt{2}-3\sqrt{3}+4\sqrt{5}}{\sqrt{2}+\sqrt{3}-\sqrt{5}}$.

21. Rationalize the denominator of $\dfrac{a}{\sqrt{a}+\sqrt[3]{b^2}}$, or $\dfrac{a}{a^{\frac{1}{2}}+b^{\frac{2}{3}}}$.

SOLUTION. — By § 134, $\sqrt{a}+\sqrt[3]{b^2}$, or $a^{\frac{1}{2}}+b^{\frac{2}{3}}$, is exactly contained in the sum of any like odd powers of $a^{\frac{1}{2}}$ and $b^{\frac{2}{3}}$, and also in the difference of any like even powers of $a^{\frac{1}{2}}$ and $b^{\frac{2}{3}}$. The lowest like powers of $a^{\frac{1}{2}}$ and $b^{\frac{2}{3}}$ that are rational numbers are the sixth powers, which are even powers. Hence, the rational expression of lowest degree in which $a^{\frac{1}{2}}+b^{\frac{2}{3}}$ is exactly contained is $(a^{\frac{1}{2}})^6-(b^{\frac{2}{3}})^6$, or a^3-b^4.

Dividing a^3-b^4 by $a^{\frac{1}{2}}+b^{\frac{2}{3}}$, the rationalizing factor for the denominator is found to be $a^{\frac{5}{2}}-a^2b^{\frac{2}{3}}+a^{\frac{3}{2}}b^{\frac{4}{3}}-ab^2+a^{\frac{1}{2}}b^{\frac{8}{3}}-b^{\frac{10}{3}}$.

Multiplying both terms of the given fraction by this factor,

$$\frac{a}{\sqrt{a}+\sqrt[3]{b^2}},\text{ or }\frac{a}{a^{\frac{1}{2}}+b^{\frac{2}{3}}}=\frac{a(a^{\frac{5}{2}}-a^2b^{\frac{2}{3}}+a^{\frac{3}{2}}b^{\frac{4}{3}}-ab^2+a^{\frac{1}{2}}b^{\frac{8}{3}}-b^{\frac{10}{3}})}{a^3-b^4}.$$

Rationalize the denominator of:

22. $\dfrac{\sqrt[3]{ab}}{\sqrt[3]{a}-\sqrt[3]{b}}$.

23. $\dfrac{2}{\sqrt[3]{x}+\sqrt{y}}$.

24. $\dfrac{\sqrt[3]{ab^2}}{\sqrt[3]{a^2}-\sqrt{b^3}}$.

25. $\dfrac{\sqrt{a}+b}{\sqrt[4]{a}-\sqrt{b}}$.

26. $\dfrac{\sqrt{ax}}{\sqrt[3]{a}-\sqrt[5]{x}}$.

27. $\dfrac{\sqrt[3]{xy^2}}{\sqrt{x}+\sqrt[5]{y}}$.

Square Root of a Binomial Quadratic Surd

356. To find the square root by inspection.

The square of a binomial may be written in the form

$$(a+b)^2=(a^2+b^2)+2\,ab.$$

Thus, $\quad(\sqrt{2}+\sqrt{6})^2=(2+6)+2\sqrt{12}=8+2\sqrt{12}.$

Therefore, the terms of the square root of $8+2\sqrt{12}$ may be obtained by separating $\sqrt{12}$ into two factors such that the sum of their squares is 8. They are $\sqrt{2}$ and $\sqrt{6}$.

$$\sqrt{8+2\sqrt{12}}=\sqrt{2}+\sqrt{6}.$$

PRINCIPLE. — *The terms of the square root of a binomial quadratic surd that is a perfect square may be obtained by dividing the irrational term by 2 and then separating the quotient into two factors, the sum of whose squares is the rational term.*

EXERCISES

357. 1. Find the square root of $14+8\sqrt{3}$.

SOLUTION

$$14+8\sqrt{3}=14+2(4\sqrt{3})=14+2\sqrt{48}.$$

Since $\quad\sqrt{48}=\sqrt{6}\times\sqrt{8}$ and $14=6+8$,

$$\sqrt{14+8\sqrt{3}}=\sqrt{6}+\sqrt{8}=\sqrt{6}+2\sqrt{2}.$$

2. Find the square root of $11-6\sqrt{2}$.

SOLUTION

$$\sqrt{11-6\sqrt{2}}=\sqrt{11-2\sqrt{18}}=\sqrt{9}-\sqrt{2}=3-\sqrt{2}.$$

Find the square root of:

3. $12+2\sqrt{35}$.
4. $16-2\sqrt{60}$.
5. $15+2\sqrt{26}$.
6. $16-2\sqrt{55}$.
7. $11+2\sqrt{30}$.
8. $7-2\sqrt{10}$.
9. $12-6\sqrt{3}$.
10. $17+12\sqrt{2}$.
11. $12+4\sqrt{5}$.
12. $11+4\sqrt{7}$.
13. $15-6\sqrt{6}$.
14. $18+6\sqrt{5}$.

Find the square root of:

15. $3-2\sqrt{2}$.

16. $6+2\sqrt{5}$.

17. $a^2+b+2a\sqrt{b}$.

18. $2a-2\sqrt{a^2-b^2}$.

358. To find the square root by using conjugate relations.

This method is useful in the more difficult cases. It depends upon the following **properties of quadratic surds.**

359. PRINCIPLE 1. — *The square root of a rational number cannot be partly rational and partly a quadratic surd.*

For, if possible, let $\sqrt{y}=\sqrt{b}\pm m$, $\sqrt{y}$ and $\sqrt{b}$ being surds.

By squaring, $y=b\pm 2m\sqrt{b}+m^2$,

and $\sqrt{b}=\pm\dfrac{y-m^2-b}{2m}$,

which is impossible, because (§ 321) a surd cannot be equal to a rational number.

Therefore, $\sqrt{y}$ cannot be equal to $\sqrt{b}\pm m$.

360. PRINCIPLE 2. — *In any equation containing rational numbers and quadratic surds, as* $a+\sqrt{b}=x+\sqrt{y}$, *the rational parts are equal, and also the irrational parts.*

Given $$a+\sqrt{b}=x+\sqrt{y}. \quad (1)$$

Since a and x are both rational, if possible, let

$$a=x\pm m. \quad (2)$$

Then, $$x\pm m+\sqrt{b}=x+\sqrt{y}, \quad (3)$$

and $$\sqrt{y}=\sqrt{b}\pm m. \quad (4)$$

Since, § 359, equation (4) is impossible, $a=x\pm m$ **is impossible**; that is, a is neither greater nor less than x.

Therefore, $a=x$, and from (1), $\sqrt{b}=\sqrt{y}$.

Hence, if $a+\sqrt{b}=x+\sqrt{y}$, $a=x$ *and* $\sqrt{b}=\sqrt{y}$.

361. Principle 3. — *If $a+\sqrt{b}$ and $a-\sqrt{b}$ are binomial quadratic surds and $\sqrt{a+\sqrt{b}}=\sqrt{x}+\sqrt{y}$, then $\sqrt{a-\sqrt{b}}=\sqrt{x}-\sqrt{y}$.*

To exclude imaginary numbers from the discussion, suppose that $a-\sqrt{b}$ is positive.

Given	$\sqrt{a+\sqrt{b}}=\sqrt{x}+\sqrt{y}$.
Squaring, § 277,	$a+\sqrt{b}=x+2\sqrt{xy}+y$.
Therefore, § 360,	$a=x+y$ and $\sqrt{b}=2\sqrt{xy}$;
whence, Ax. 2,	$a-\sqrt{b}=x+y-2\sqrt{xy}$.
Hence, § 289,	$\sqrt{a-\sqrt{b}}=\sqrt{x}-\sqrt{y}$.

EXERCISES

362. 1. Find the square root of $21+6\sqrt{10}$.

Solution

Let	$\sqrt{x}+\sqrt{y}=\sqrt{21+6\sqrt{10}}$.	(1)
Then, § 361,	$\sqrt{x}-\sqrt{y}=\sqrt{21-6\sqrt{10}}$.	(2)
Multiplying (1) by (2),	$x-y=\sqrt{441-360}=\sqrt{81}$,	
or	$x-y=9$.	(3)
Squaring (1), § 277,	$x+2\sqrt{xy}+y=21+6\sqrt{10}$.	
Therefore, § 360,	$x+y=21$.	(4)
Solving (4) and (3),	$x=15$, $y=6$.	
	$\therefore \sqrt{x}=\sqrt{15}$, $\sqrt{y}=\sqrt{6}$.	
Hence, from (1),	$\sqrt{21+6\sqrt{10}}=\sqrt{15}+\sqrt{6}$.	

Find the square root of:

2. $25+10\sqrt{6}$.	8. $16+6\sqrt{7}$.	14. $2+\sqrt{3}$.
3. $19+6\sqrt{2}$.	9. $21-8\sqrt{5}$.	15. $6+\sqrt{35}$.
4. $45+30\sqrt{2}$.	10. $47-12\sqrt{11}$.	16. $1+\frac{2}{3}\sqrt{2}$.
5. $35-14\sqrt{6}$.	11. $56+32\sqrt{3}$.	17. $2+\frac{4}{5}\sqrt{6}$.
6. $11+6\sqrt{2}$.	12. $35-12\sqrt{6}$.	18. $30+20\sqrt{2}$.
7. $24-8\sqrt{5}$.	13. $56-12\sqrt{3}$.	19. $18-6\sqrt{5}$.

RADICAL EQUATIONS

363. An equation involving an irrational root of an unknown number is called an **irrational**, or **radical, equation.**

$x^{\frac{1}{2}} = 3$, $\sqrt{x+1} = \sqrt{x-4} + 1$, and $\sqrt[3]{x-1} = 2$ are radical equations.

364. A radical equation may be freed of radicals, wholly or in part, by raising both members, suitably prepared, to the same power. If the given equation contains more than one radical, involution may have to be repeated.

When the following equations have been freed of radicals, the resulting equations will be found to be simple equations. Other varieties of radical equations are treated subsequently.

EXERCISES

365. 1. Given $\sqrt{2x} + 4 = 10$, to find the value of x.

SOLUTION

$$\sqrt{2x} + 4 = 10.$$

Transposing, $\sqrt{2x} = 6.$

Squaring, $2x = 36.$

$\therefore x = 18.$

VERIFICATION. — Substituting 18 for x in the given equation and (§ 317) considering only the positive value of $\sqrt{2x}$, we have $\sqrt{36} + 4 = 10$; that is, $10 = 10$, an identity; hence, the equation is satisfied for $x = 18$.

2. Given $\sqrt{x-7} + \sqrt{x} = 7$, to find the value of x.

SOLUTION

$$\sqrt{x-7} + \sqrt{x} = 7.$$

Transposing, $\sqrt{x-7} = 7 - \sqrt{x}.$

Squaring, $x - 7 = 49 - 14\sqrt{x} + x.$

Transposing and combining, $14\sqrt{x} = 56.$

Dividing by 14, $\sqrt{x} = 4.$

Squaring, $x = 16.$

VERIFICATION. $\sqrt{16-7} + \sqrt{16} = \sqrt{9} + \sqrt{16} = 3 + 4 = 7$; that is, $7 = 7$.

3. Given $\sqrt{14+\sqrt{1+\sqrt{x+8}}}=4$, to find the value of x.

SOLUTION

$$\sqrt{14+\sqrt{1+\sqrt{x+8}}}=4.$$

Squaring, $\quad 14+\sqrt{1+\sqrt{x+8}}=16.$

Transposing, etc., $\quad \sqrt{1+\sqrt{x+8}}=16-14=2.$

Squaring, $\quad 1+\sqrt{x+8}=4.$

Transposing, etc., $\quad \sqrt{x+8}=4-1=3.$

Squaring, $\quad x+8=9.$

$$\therefore x=9-8=1.$$

VERIFICATION. $\sqrt{14+\sqrt{1+\sqrt{1+8}}}=\sqrt{14+\sqrt{1+3}}$

$$=\sqrt{14+2}=4\text{; that is, } 4=4.$$

General Directions. — *Transpose so that the radical term, if there is but one, or the most complex radical term, if there is more than one, may constitute one member of the equation.*

Then raise each member to a power corresponding to the order of that radical and simplify.

If the equation is not freed of radicals by the first involution, proceed again as at first.

Solve, and verify each result:

4. $\sqrt{x+11}=4.$

5. $\sqrt{x+5}=31.$

6. $\sqrt{x-a^2}=b.$

7. $\sqrt[3]{x-1}=2.$

8. $\sqrt[3]{x-a^3}=a.$

9. $\sqrt[3]{x}+b=a.$

10. $1+\sqrt{x}=5.$

11. $1+2\sqrt{x}=7-\sqrt{x}.$

12. $\sqrt{x+16}-\sqrt{x}=2.$

13. $\sqrt{2x}-\sqrt{2x-15}=1.$

14. $\sqrt{x^2+x+1}=2-x.$

15. $3\sqrt{x^2-9}=3x-3.$

16. $\sqrt{x}+2=\sqrt{x+32}.$

17. $5-\sqrt{x+5}=\sqrt{x}.$

Solve, and verify each result:

18. $\sqrt{x^2-5x+7}+2=x$. **19.** $\sqrt{9x+8}+\sqrt{9x}-4=0$.

20. $4-\sqrt{4-8x+9x^2}=3x$.

21. $\sqrt{2(1-x)(3-2x)}-1=2x$.

22. $\sqrt{2x-1}+\sqrt{2x+4}=5$.

23. $\sqrt{3x-5}+\sqrt{3x+7}=6$.

24. $\sqrt{16x+3}+\sqrt{16x+8}=5$.

25. $\sqrt{1+x\sqrt{x^2+12}}=1+x$.

26. $\sqrt{7+3\sqrt{5x-16}}-4=0$.

27. $2x-\sqrt{4x^2-\sqrt{16x^2-7}}=1$.

28. $2\sqrt{x}-\sqrt{4x-22}-\sqrt{2}=0$.

29. $\sqrt{2(x+1)}+\sqrt{2x-1}=\sqrt{8x+1}$.

30. $\sqrt{3x+7}+\sqrt{4x-3}=\sqrt{4x+4}+\sqrt{3x}$.

31. $\sqrt{\sqrt{\sqrt{2x+56}}}=2$.

32. $\sqrt{7+\sqrt{1+\sqrt{4+\sqrt{1+2\sqrt{x}}}}}=3$.

33. Solve the equation $\dfrac{5}{\sqrt{3x+2}}=\sqrt{3x+2}+\sqrt{3x-1}$.

Suggestion. — Clear the equation of fractions.

34. Solve the equation $\dfrac{\sqrt{3x}+15}{\sqrt{3x}+5}=\dfrac{\sqrt{3x}+6}{\sqrt{3x}+1}$.

Suggestion. — Some labor may be saved by reducing each fraction to a mixed number and *simplifying before clearing of fractions.*

Thus,
$$1+\frac{10}{\sqrt{3x}+5}=1+\frac{5}{\sqrt{3x}+1}.$$

Canceling, and dividing both members by 5,
$$\frac{2}{\sqrt{3x}+5}=\frac{1}{\sqrt{3x}+1}.$$

Solve and verify:

35. $\dfrac{\sqrt{2x}+9}{\sqrt{2x}-7}=\dfrac{\sqrt{2x}+20}{\sqrt{2x}-12}.$

36. $\dfrac{\sqrt{x}+18}{\sqrt{x}+2}=\dfrac{32}{\sqrt{x}+6}+1.$

37. $\dfrac{\sqrt{s-1}}{\sqrt{s+5}}=\dfrac{\sqrt{s-3}}{\sqrt{s-1}}.$

38. $\dfrac{\sqrt{v}-6}{\sqrt{v}-1}=\dfrac{\sqrt{v}-8}{\sqrt{v}-5}.$

39. $\dfrac{\sqrt{t-3}}{\sqrt{t+1}}=\dfrac{\sqrt{t-4}}{\sqrt{t-2}}.$

40. $\dfrac{\sqrt{2r}+6}{\sqrt{2r}+4}=\dfrac{\sqrt{2r}+2}{\sqrt{2r}+1}.$

41. $\dfrac{\sqrt{11n}+\sqrt{2n+3}}{\sqrt{11n}-\sqrt{2n+3}}=\dfrac{8}{3}.$

42. $\dfrac{2\sqrt{2x}+4}{2\sqrt{2x}-4}=\dfrac{3\sqrt{x+1}+9}{3\sqrt{x+1}-9}.$

43. $\dfrac{\sqrt{m+1}-\sqrt{m-1}}{\sqrt{m+1}+\sqrt{m-1}}=\dfrac{1}{2}.$

44. $\dfrac{\sqrt{4z+3}+2\sqrt{z-1}}{\sqrt{4z+3}-2\sqrt{z-1}}=5.$

45. $\dfrac{\sqrt{\sqrt{5x}-9}}{\sqrt{\sqrt{5x}+11}}=\dfrac{\sqrt{\sqrt{5x}-21}}{\sqrt{\sqrt{5x}-16}}.$

SUGGESTION. — First square both members.

46. $\dfrac{x-3}{\sqrt{x}-\sqrt{3}}=\dfrac{\sqrt{x}+\sqrt{3}}{2}+2\sqrt{3}.$

SUGGESTION. — Begin by simplifying the first member.

47. $\sqrt{2x}-\sqrt{2x-7}=\dfrac{3}{\sqrt{2x-7}}.$

48. Solve $\dfrac{\sqrt{x+a}+\sqrt{x-a}}{\sqrt{x+a}-\sqrt{x-a}}=2+\dfrac{\sqrt{x^2-a^2}}{a}$ for x.

SUGGESTION. — Rationalize the denominator of the first fraction.

Solve for x, and verify:

49. $\sqrt{x}+\sqrt{x-(a-b)^2}=a+b.$

50. $a\sqrt{x}-b\sqrt{x}=a^2+b^2-2ab.$

51. $\sqrt{5ax-9a^2}+a=\sqrt{5ax}.$

52. $\sqrt{x+3a}=\dfrac{6a}{\sqrt{x+3a}}-\sqrt{x}.$

53. Solve $\sqrt{x}+\sqrt{2x}+\sqrt{3x}=\sqrt{a}$ for x.

SOLUTION

$$\sqrt{x}+\sqrt{2x}+\sqrt{3x}=\sqrt{a}. \qquad (1)$$

Factoring, $$\sqrt{x}(1+\sqrt{2}+\sqrt{3})=\sqrt{a}. \qquad (2)$$

$$\therefore \sqrt{x}=\frac{\sqrt{a}}{1+\sqrt{2}+\sqrt{3}}$$

$$=\frac{\sqrt{a}(1+\sqrt{2}-\sqrt{3})}{(1+\sqrt{2}+\sqrt{3})(1+\sqrt{2}-\sqrt{3})}$$

$$=\frac{\sqrt{a}(1+\sqrt{2}-\sqrt{3})}{2\sqrt{2}}. \qquad (3)$$

Squaring, $$x=\frac{a}{8}(1+\sqrt{2}-\sqrt{3})^2. \qquad (4)$$

Solve for x:

54. $\sqrt{2x}+\sqrt{3x}+\sqrt{5x}=\sqrt{m}$.

55. $\sqrt{2x}+\sqrt{3x}-\sqrt{5x}=\sqrt{c}$.

56. $\sqrt{x-a}+\sqrt{2(x-a)}=\sqrt{3x+a\sqrt{2}}$.

366. From §§ 364, 365, the student will have observed that radical equations are freed of radicals either by *rationalization* or by *involution*.

Thus,	$\sqrt{2x}-6=0$ (1)	$\sqrt{2x}+6=0$ (2)
Multiplying by	$\sqrt{2x}+6$	$\sqrt{2x}-6$
	$2x-36=0$	$2x-36=0$
	$\therefore x=18$	$\therefore x=18$

If the positive, or principal, square root of $2x$ is taken, $x=18$ satisfies (1) but not (2); if the negative square root of $2x$ is taken, $x=18$ satisfies (2) but not (1).

It has been agreed, however, that the sign $\sqrt{\ }$ shall denote only principal roots in this chapter, and because of this *arbitrary convention*, our conclusion must be that (1) has the root $x=18$ and that (2) has no root, or is impossible.

According to this view, when both members of (1) are multiplied by $\sqrt{2x}+6$, no root is introduced because $\sqrt{2x}+6=0$ has no root; but when both members of (2), which has no root, are multiplied by $\sqrt{2x}-6$, the root of $\sqrt{2x}-6=0$, which is $x=18$, is introduced (§ 230).

A root may be introduced in this way by *rationalization*, or by the equivalent process of *squaring*.

Thus, $\qquad \sqrt{2x}+6=0. \qquad (2)$

Transposing, $\qquad \sqrt{2x}=-6.$

Squaring, § 277, $\qquad 2x=36.$

$\therefore x=18.$

Verifying, $\qquad \sqrt{2\cdot 18}+6=6+6\neq 0.$

The symbol $\neq$ is read 'is not equal to.'

EXERCISES

367. 1. Solve, if possible, the equation $\sqrt{x-7}-\sqrt{x}=7$.

SOLUTION. — Transposing, squaring, simplifying, etc.,

$$\sqrt{x}=-4.$$

Squaring, $\qquad x=16.$

VERIFICATION. $\quad \sqrt{16-7}-\sqrt{16}=\sqrt{9}-\sqrt{16}=3-4\neq 7.$

Hence, the equation has no root, or is *impossible*.

Solve, and verify to discover which of the following equations are impossible; then change these to true equations:

2. $\sqrt{2x}+\sqrt{2x-3}=1.$

3. $\sqrt{3x+7}+\sqrt{3x}=7.$

4. $2\sqrt{x}+\sqrt{4x-11}=1.$

5. $\sqrt{4x+5}-2\sqrt{x-1}=9.$

6. $\sqrt{4x}-\sqrt{x}=\sqrt{9x-32}.$

7. $\sqrt{5x-1}-1=\sqrt{5x+16}.$

8. $\sqrt{x+1}+\sqrt{x+2}-\sqrt{4x+5}=0.$

9. $\sqrt{2(x^2+3x-5)}=(x+2)\sqrt{2}.$

10. $\dfrac{\sqrt{x-5}}{\sqrt{x-4}}+\dfrac{\sqrt{x+1}}{\sqrt{x+8}}=0.$

11. $\dfrac{\sqrt{19x}+\sqrt{2x+11}}{\sqrt{19x}-\sqrt{2x+11}}=2\frac{1}{8}.$

IMAGINARY NUMBERS

368. Our number system now comprises **natural numbers**, 1, 2, 3, ...; **fractions**, arising from the indicated division of one natural number by another; **negative numbers** (denoting opposition to positive numbers), arising from the subtraction of a number from a less number; **surds**, arising from the attempt to extract a root of a number that is not a perfect power; and finally **imaginary numbers**, arising from the attempt to extract an even root of a negative number (§ 285).

In this chapter only imaginary numbers of the second order will be treated.

Before the introduction of imaginary numbers, the only numbers known were those *whose squares are positive*, now called **real numbers** to distinguish them from **imaginary numbers**, *whose squares are negative.*

369. Since the square of an imaginary number is negative, imaginary numbers present an apparent exception, ***in regard to signs***, to the distributive law for evolution. Apparently

$$\sqrt{-1} \times \sqrt{-1} \text{ would equal } \sqrt{(-1)(-1)} = \sqrt{+1} = \pm 1.$$

But by the definition of a root, the square of the square root of a number is the number itself.

$$\text{Hence,} \quad \sqrt{-1} \times \sqrt{-1} = (\sqrt{-1})^2 = -1, \text{ not } +1. \qquad (A)$$

In this chapter it will be assumed that imaginary numbers obey the same laws as real numbers, the signs being *determined by (A)*, which we call the **fundamental property of imaginaries.**

370. Powers of $\sqrt{-1}$.

$$(\sqrt{-1}) \qquad\qquad = +\sqrt{-1};$$

$$(\sqrt{-1})^2 = (\sqrt{-1})(\sqrt{-1}) \qquad\qquad = -1;$$

$$(\sqrt{-1})^3 = (\sqrt{-1})^2\sqrt{-1} = (-1)\sqrt{-1} = -\sqrt{-1};$$

$$(\sqrt{-1})^4 = (\sqrt{-1})^2(\sqrt{-1})^2 = (-1)(-1) = +1;$$

$$(\sqrt{-1})^5 = (\sqrt{-1})^4\sqrt{-1} = (+1)\sqrt{-1} = +\sqrt{-1};$$

and so on. Hence, if $n = 0$ or a positive integer,

$$\left.\begin{array}{l}(\sqrt{-1})^{4n+1} = +\sqrt{-1};\ (\sqrt{-1})^{4n+2} = -1;\\ (\sqrt{-1})^{4n+3} = -\sqrt{-1};\ (\sqrt{-1})^{4n+4} = +1.\end{array}\right\} \qquad (B)$$

Hence, *any even power of $\sqrt{-1}$ is real and any odd power is imaginary.*

For brevity $\sqrt{-1}$ is often written i.

371. Operations involving imaginary numbers.

EXERCISES

Find the value of:

1. $(\sqrt{-1})^6$. 3. $(\sqrt{-1})^{10}$. 5. $(\sqrt{-1})^{18}$. 7. $(-i)^3$.

2. $(\sqrt{-1})^7$. 4. $(\sqrt{-1})^{21}$. 6. $(\sqrt{-1})^{15}$. 8. $(-i)^8$.

9. Add $\sqrt{-a^4}$ and $\sqrt{-16a^4}$.

SOLUTION

$$\sqrt{-a^4} + \sqrt{-16a^4} = a^2\sqrt{-1} + 4a^2\sqrt{-1} = 5a^2\sqrt{-1}.$$

Simplify:

10. $\sqrt{-4} + \sqrt{-49}$. 13. $\sqrt{-12} + 4\sqrt{-3}$.

11. $\sqrt{-9} + \sqrt{-64}$. 14. $5\sqrt{-18} - \sqrt{-72}$.

12. $2\sqrt{-4} + 3\sqrt{-1}$. 15. $3\sqrt{-20} - \sqrt{-80}$.

16. $\sqrt{-16a^2x^2} + \sqrt{-a^2x^2} - \sqrt{-9a^2x^2}$.

17. $(\sqrt{-a} + 3\sqrt{-b}) + (\sqrt{-a} - 3\sqrt{-b})$.

18. $(\sqrt{-9xy} - \sqrt{-xy}) - (\sqrt{-4xy} + \sqrt{-xy})$.

19. $\sqrt{-x^2} + \sqrt{-4x^2} - \sqrt{-x^3} + 3x\sqrt{-x}$.

20. $\sqrt{-16} - 3\sqrt{-4} + \sqrt{-18} + \sqrt{-50} + \sqrt{-25}$.

21. $\sqrt{-8} + a\sqrt{-2} - \sqrt{-98} - 5\sqrt{-2a^2}$.

22. $\sqrt{1-5} - 3\sqrt{1-10} + 2\sqrt{5-30}$.

23. Multiply $3\sqrt{-10}$ by $2\sqrt{-8}$.

PROCESS

$$\begin{aligned} 3\sqrt{-10} \times 2\sqrt{-8} &= 3\sqrt{10}\sqrt{-1} \times 2\sqrt{8}\sqrt{-1} \\ &= 6\sqrt{10 \times 8} \times (-1) \\ &= -6\sqrt{80} = -24\sqrt{5} \end{aligned}$$

EXPLANATION. — To determine the sign of the product, each imaginary number is reduced to the form $b\sqrt{-1}$. The numbers are then multiplied as ordinary radicals, subject to (A), § 369, that $\sqrt{-1} \times \sqrt{-1} = -1$.

24. Multiply $\sqrt{-2} + 3\sqrt{-3}$ by $4\sqrt{-2} - \sqrt{-3}$.

FIRST SOLUTION

$$\begin{aligned} &\sqrt{-2} + 3\sqrt{-3} = (\sqrt{2} + 3\sqrt{3})\sqrt{-1}, \\ &4\sqrt{-2} - \sqrt{-3} = (4\sqrt{2} - \sqrt{3})\sqrt{-1}; \\ &\therefore (\sqrt{-2} + 3\sqrt{-3})(4\sqrt{-2} - \sqrt{-3}) \\ &= (\sqrt{2} + 3\sqrt{3})(4\sqrt{2} - \sqrt{3})(\sqrt{-1})^2 \\ &= (8 + 12\sqrt{6} - \sqrt{6} - 9)(-1) \\ &= 1 - 11\sqrt{6}. \end{aligned}$$

SECOND SOLUTION

$$\begin{array}{l} \sqrt{-2} + 3\sqrt{-3} \\ 4\sqrt{-2} - \sqrt{-3} \\ \hline -4\sqrt{4} - 12\sqrt{6} \\ +3\sqrt{9} + \sqrt{6} \\ \hline 1 \quad - 11\sqrt{6} \end{array}$$

Multiply:

25. $3\sqrt{-5}$ by $2\sqrt{-15}$.

26. $4\sqrt{-27}$ by $\sqrt{-12}$.

27. $2\sqrt{-8}$ by $5\sqrt{-3}$.

28. $8\sqrt{-1}$ by $\sqrt{-b^8}$.

29. $\sqrt{-125}$ by $\sqrt{-108}$.

30. $\sqrt{-100}$ by $\sqrt{-30}$.

$\sqrt{-6} + \sqrt{-3}$ by $\sqrt{-6} - \sqrt{-3}$.

$\sqrt{-ab} + \sqrt{-a}$ by $\sqrt{-ab} - \sqrt{-a}$.

$\sqrt{-xy} + \sqrt{-x}$ by $\sqrt{-xy} + \sqrt{-x}$.

$\sqrt{-50} - \sqrt{-12}$ by $\sqrt{-8} - \sqrt{-75}$.

$\sqrt{-a} + \sqrt{-b} + \sqrt{-c}$ by $\sqrt{-a} + \sqrt{-b} - \sqrt{-c}$.

Divide $\sqrt{-12}$ by $\sqrt{-3}$.

Solution

$$\frac{\sqrt{-12}}{\sqrt{-3}} = \frac{\sqrt{12}\sqrt{-1}}{\sqrt{3}\sqrt{-1}} = \frac{\sqrt{12}}{\sqrt{3}} = \sqrt{4} = 2.$$

Divide $\sqrt{12}$ by $\sqrt{-3}$.

Solution

$$\frac{\sqrt{12}}{\sqrt{-3}} = \frac{\sqrt{12}}{\sqrt{3}\sqrt{-1}} = \frac{\sqrt{4}}{\sqrt{-1}} = \frac{2}{\sqrt{-1}}$$

$$= \frac{2\sqrt{-1}}{-1} = -2\sqrt{-1}.$$

Divide 5 by $(\sqrt{-1})^3$.

Solution

$$\frac{5}{(\sqrt{-1})^3} = \frac{5(+1)}{(\sqrt{-1})^3} = \frac{5(\sqrt{-1})^4}{(\sqrt{-1})^3} = 5\sqrt{-1}.$$

ride:

$\sqrt{-18}$ by $\sqrt{-3}$.

$\sqrt{27}$ by $\sqrt{-3}$.

$14\sqrt{-5}$ by $2\sqrt{-7}$.

$-\sqrt{-a^2}$ by $\sqrt{-b^2}$.

1 by $\sqrt{-1}$.

$\sqrt{8} + 3\sqrt{14}$ by $\sqrt{-2}$.

$\sqrt{12} + \sqrt{3}$ by $\sqrt{-3}$.

46. -2 by $\sqrt{-1}$.

47. $(\sqrt{-1})^5$ by $\frac{1}{3}\sqrt{-1}$.

48. $(\sqrt{-1})^3$ by $(\sqrt{-1})^{15}$.

49. $\sqrt{4ab}$ by $\sqrt{-bc}$.

50. $(\sqrt{-1})^{14}$ by $-\frac{1}{2}\sqrt{-1}$.

51. $(\sqrt{-1})^{10}$ by $(\sqrt{-1})^{-2}$.

52. $\sqrt{-a^2} + b\sqrt{-1}$ by $\sqrt{-ab}$.

53. $\sqrt{-4}$ by $\sqrt{-2} \cdot \sqrt{-2} \cdot \sqrt{-1}$.

REVIEW

372. 1. Distinguish between an equation and an identity; between an integral and a fractional equation. Illustrate.

2. When is a literal equation an identity?

3. State what is meant by a graph. Of what practical use are graphs?

4. Define abscissa; ordinate; coördinates. Interpret the equation $A = (-4, 3)$.

5. Tell how to determine where a graph crosses the x-axis; the y-axis.

6. Construct the graph of $2y = 3x - 4$.

7. Why are simple equations sometimes called linear equations?

8. State the law of signs for involution; the law of exponents.

9. How may the involution of a trinomial be performed by the use of the binomial theorem?

10. Tell how the 12th root of an expression may be obtained.

11. Define root of an equation; equivalent equations; simultaneous equations; independent equations; indeterminate equations; elimination of an unknown number.

12. Upon what axiom is elimination by addition based? elimination by comparison?

13. Define radical; radicand; surd; conjugate surds. Illustrate each. Is $\sqrt{2+\sqrt{4}}$ a surd? State reasons for your answer.

14. What are similar radicals? Illustrate.

15. Tell what is meant by the principal root of a number. *What is the* principal square root of 4*?* the principal cube *root of* -8*?*

16. Represent $\sqrt{10}$ inches exactly by a line.

17. Show the difference in meaning between $(a^b)^c$ and $a^b \times a^c$.

18. What does the numerator of a fractional exponent indicate? the denominator?

19. Show that $x^0 = 1$; that $\frac{a^5}{b^{-5}} = a^5b^5$.

20. Factor $a^{-2} + 2a^{-1}b^{-1} + b^{-2}$, giving reasons for each step.

21. Define and illustrate mixed surd; entire surd. Tell how to reduce a mixed surd to an entire surd.

22. What is a radical equation? Give the steps in the solution of such an equation.

23. Define real number; imaginary number; rational number; irrational number.

24. Classify the following numbers as real or imaginary; as rational or irrational:

$$2,\ \sqrt{4},\ \sqrt{2},\ \sqrt{5},\ \sqrt{-2},\ \sqrt{-5},\ \sqrt{a^3},\ \sqrt[3]{a^6},\ \sqrt[4]{-a},$$

a being a positive number.

25. Illustrate how, in finding the value of an expression with an irrational denominator, it is advantageous to rationalize the denominator first.

26. Find the value of i^2, i^5, i^9, i^8. What is the value of even powers of i? of odd powers of i?

27. Solve graphically the simultaneous equations

$$\begin{cases} 2x - 3y = 10, \\ 5x + 2y = 6. \end{cases}$$

28. If a system of two linear equations is indeterminate, how will the fact be shown by the graphs of the equations, referred to the same axes? how, if they are inconsistent?

29. When a is positive, is $\sqrt{-a}$ real or imaginary? When a is negative, is $\sqrt{-a}$ real or imaginary?

30. Find the sum and the product of $2\sqrt{-4}$ and $3\sqrt{-9}$.

31. Subtract $\sqrt{-9}$ from $\sqrt{-81}$; divide $\sqrt{-81}$ by $\sqrt{-9}$.

EXERCISES

373. Reduce to simplest form:

1. $\dfrac{6x^3-7x^2-5x}{9x^3-25x}$.

2. $\dfrac{8x^2+18x-5}{12x^2+5x-2}$.

3. $\dfrac{a^2x^2-a\sqrt{x}+x}{\sqrt{x}}$.

4. $\dfrac{a^2-2a\sqrt{b}+b}{a-\sqrt{b}}$.

5. $\dfrac{\sqrt{2}-\sqrt{3}-\sqrt{5}}{\sqrt{2}+\sqrt{3}+\sqrt{5}}$.

6. $\dfrac{2-\sqrt{5}}{2+\sqrt{5}}+\dfrac{2\sqrt{3}}{\sqrt{243}}$.

7. $\dfrac{x-y}{x+y}-\dfrac{y+x}{y-x}-\dfrac{4x^2y^2}{x^4-y^4}$.

8. $\dfrac{\sqrt{2}-\sqrt{3}}{\sqrt{2}+\sqrt{3}}-\dfrac{\sqrt{2}+\sqrt{3}}{\sqrt{2}-\sqrt{3}}$.

9. $\dfrac{x+\sqrt{xy}+y}{\sqrt{x}+\sqrt{y}}+\dfrac{x\sqrt{x}+y\sqrt{y}}{x+y}$.

10. $\dfrac{1}{\sqrt{a}+\sqrt{b}}-1+\dfrac{1}{\sqrt{a}-\sqrt{b}}$.

11. $\dfrac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}$.

12. $\dfrac{1}{1-\sqrt{2x}}+\dfrac{1}{1+\sqrt{2x}}+\dfrac{1}{1-2x}$.

13. $\dfrac{x+\sqrt{x^2-a^2}}{x-\sqrt{x^2-a^2}}-\dfrac{x-\sqrt{x^2-a^2}}{x+\sqrt{x^2-a^2}}$.

14. $\dfrac{\sqrt{a+1}+\sqrt{a-1}}{\sqrt{a+1}-\sqrt{a-1}}-\dfrac{\sqrt{a+1}-\sqrt{a-1}}{\sqrt{a+1}+\sqrt{a-1}}$.

15. $\dfrac{a^2-b}{a^2-2a\sqrt{b}+b}\times\dfrac{a^2-4a\sqrt{b}+4b}{a^2+2a\sqrt{b}+b}$.

16. $\left(1-\dfrac{a}{b}\right)\div\left(1+\sqrt{\dfrac{a}{b}}\right)$.

17. $\dfrac{\dfrac{1+a+a^2}{1+\sqrt{a}+a}}{\dfrac{1-\sqrt{a}+a}{1-\sqrt{a}}}$.

18. $1\div\left(\sqrt{\dfrac{a}{b}}-\sqrt{\dfrac{b}{a}}\right)+\dfrac{\sqrt{ab}}{a+b}$.

19. $\dfrac{\left(\dfrac{a}{\sqrt{x}}+\dfrac{\sqrt{x}}{a}\right)\left(\dfrac{a}{\sqrt{x}}-\dfrac{\sqrt{x}}{a}\right)}{\left(\dfrac{a}{\sqrt{x}}-\dfrac{\sqrt{x}}{a}\right)\left(\dfrac{a}{\sqrt{x}}-\dfrac{\sqrt{x}}{a}\right)}$.

pand:

. $(a^3-b^2)^3$.

24. $(a^{-2}+a^{-1})^2$.

25. $(a^{-1}+b)^4$.

26. $(a^{\frac{1}{2}}-b^{\frac{1}{2}})^6$.

27. $(a^{\frac{1}{2}}-b^{-\frac{1}{2}})^4$.

28. $(a^{-\frac{1}{3}}-b^{-\frac{1}{3}})^6$.

29. $(a^{\frac{1}{3}}+b^{\frac{1}{2}})^6$.

. $(2a-3b)^4$.

. $\left(\frac{a}{3}-\frac{b}{2}\right)^3$.

. $\left(ax+\frac{1}{a}\right)^5$.

30. $(a-\sqrt{b})^4$.

31. $(\sqrt{x}+\sqrt{y})^6$.

32. $(\sqrt{2}-\sqrt{3})^4$.

33. $(\sqrt{5}-2)^6$.

34. $(\sqrt[3]{4}-\sqrt[3]{2})^3$.

35. $(\sqrt{2}-\sqrt[3]{2})^6$.

xtract the square root of:

36. $\frac{9x^4}{4}+3x^3-x^2-\frac{4x}{3}+\frac{4}{9}$.

37. $\frac{x^2}{4}+4y^2+\frac{z^2}{16}-2xy+\frac{xz}{4}-yz$.

38. $a^2+12a^{\frac{3}{2}}b^{\frac{1}{2}}+54ab+108a^{\frac{1}{2}}b^{\frac{3}{2}}+81b^2$.

39. $1+2\sqrt{x}-x-2x\sqrt{x}+x^2$.

40. $a+4b+9c-4\sqrt{ab}+6\sqrt{ac}-12\sqrt{bc}$.

41. $x^2-4x\sqrt{xy}+6xy-4y\sqrt{xy}+y^2$.

ind the square root of:

2. 81234169.

3. 64064016.

4. .00022801.

5. .1 to four places.

46. $56+14\sqrt{15}$.

47. $47-12\sqrt{15}$.

48. $62+20\sqrt{6}$.

49. $51-36\sqrt{2}$.

xtract the cube root of:

50. $x^3-9x+27x^{-1}-27x^{-3}$.

51. $27x^3+27x^2-5+\frac{1}{3x^2}-\frac{1}{27x^3}$.

52. $x^3+3x^2\sqrt{x}-5x\sqrt{x}+3\sqrt{x}-1$.

53. Find the cube root of $2\sqrt{2} - 6\sqrt[3]{2} + 3\sqrt{2}\sqrt[3]{4} - 2$.

54. Extract the cube root of 510,082,399.

55. Extract the cube root of 1,042,590,744.

56. Extract the cube root of 2 to three decimal places.

57. Find the first four terms of $\sqrt{1 + x - x^2}$.

58. Find the first three terms of $\sqrt[3]{1 + x^3}$.

59. Find the fourth root of

$$a^6 - 4\,a^4\sqrt{ab^{-1}} + 6\,a^3b^{-1} - 4\,ab^{-1}\sqrt{ab^{-1}} + b^{-2}.$$

60. Find the sixth root of

$$8 - 48\sqrt{a} + 120\,a - 160\,a\sqrt{a} + 120\,a^2 - 48\,a^2\sqrt{a} + \text{[illegible]}$$

If $a^m \times a^n = a^{m+n}$ for all values of m and n, show that:

61. $a^{-2} = \dfrac{1}{a^2}$.

62. $a^{\frac{3}{2}} = \sqrt{a^3} = (\sqrt{a})^3$.

63. $2\,a^{-\frac{1}{3}} = \dfrac{2\sqrt[3]{a^2}}{a}$.

64. $(ab)^0 = 1$.

65. $(abc)^3 = a^3b^3c^3$.

66. $\left(\dfrac{a}{b}\right)^3 = \dfrac{a^3}{b^3}$.

Find the value of:

67. $16^{\frac{3}{4}}$.

68. $27^{\frac{2}{3}}$.

69. $8^{-\frac{2}{3}}$.

70. $(a^4x^4)^{\frac{3}{2}}$.

71. $(b^2y^4)^{-\frac{3}{2}}$.

72. $(a^nb^n)^{-\frac{1}{n}}$.

73. $(\frac{32}{243})^{-\frac{3}{5}}$.

74. $(\frac{36}{49})^{-\frac{3}{2}}$.

75. $(-\frac{8}{27})^{-\frac{2}{3}}$.

76. For what values of n is $(a - b)^n = (b - a)^n$?

Simplify, expressing results with positive exponents:

77. $(36\,a^{-3} \div 25\,a^{-2})^{-\frac{1}{2}}$.

78. $(8\,a^3x^6 \times 64\,a^{-4}x^{-5})^{-\frac{1}{3}}$.

79. $(a^{\frac{1}{2}}b^{\frac{1}{4}})^{\frac{4}{3}} \div (a^{\frac{1}{6}}b^{\frac{1}{4}})^2$.

80. $(\sqrt{a^3x^{-3}} \div \sqrt[3]{a^2x^{-2}})^{\text{[illegible]}}$.

81. $(\sqrt{a^{-1}b^4} \div \sqrt{a^2b})^{-\frac{1}{2}}$.

82. $(\sqrt{a} \div \sqrt[3]{a}) \div \sqrt[4]{a}$.

83. $\dfrac{a-b}{a^{\frac{1}{3}}-b^{\frac{1}{3}}}-\dfrac{a+b}{a^{\frac{1}{3}}+b^{\frac{1}{3}}}+\dfrac{2\,a^{\frac{1}{3}}b}{a^{\frac{2}{3}}-b^{\frac{2}{3}}}$.

84. $\dfrac{1+a^{-1}b}{1-a^{-1}b}\div\left(\dfrac{1+ab^{-1}+a^2b^{-2}}{1-ab^{-1}+a^2b^{-2}}\times\dfrac{1+a^{-3}b^3}{1-a^{-3}b^3}\right)$.

Solve the following equations:

85. $\dfrac{x+1}{x-1}-\dfrac{x-1}{x+1}=\dfrac{3-5\,x}{1-x^2}$.

86. $\dfrac{7-2\,x}{10}-\dfrac{2\,x-1}{5}+\dfrac{x}{2}=\dfrac{5\,x-6.2}{2\,x}-\dfrac{17+3\,x}{30}$.

87. $\dfrac{4\,x-17}{9}-\dfrac{3\frac{2}{3}-22\,x}{33}=x-\dfrac{6}{x}\left(1-\dfrac{x^2}{54}\right)$.

88. $\begin{cases}\dfrac{3\,x-5\,y}{3}-\dfrac{2\,x-8\,y-9}{12}=\dfrac{y}{2}+\dfrac{7}{12},\\[2ex] \dfrac{7}{2}\left(\dfrac{x}{7}+\dfrac{y}{4}+1\frac{1}{3}\right)-\dfrac{10}{3}\left(4\,x-\dfrac{y}{8}-24\right)=0.\end{cases}$

89. $\begin{cases}3\,x+1=2\,y,\\ (x+5)(y+7)=(x+1)(y-9)+112.\end{cases}$

Simplify, expressing results with positive exponents:

90. $\dfrac{\left(\dfrac{a}{27}\div\dfrac{a^{-2}}{8}\right)^{-\frac{2}{3}}-x^2}{\dfrac{3\,a^{-1}+2\,x}{2}}$.

91. $\{a^{-2}[a^{\frac{3}{5}}(a^{\frac{3}{5}})^{\frac{1}{3}}]^5\}^{\frac{5}{2}}$.

92. $\left[\dfrac{\left(\dfrac{a^3b}{x^2y}\right)^{\frac{1}{2}}}{\left(\dfrac{a^3b^2}{xy^2}\right)^{\frac{1}{3}}}\right]^6-\dfrac{(ax^{-1})^8}{b}$.

93. $\left[\dfrac{x^{-\frac{1}{2}}y^{-\frac{2}{3}}}{x^{-\frac{1}{6}}y^{-1}}\div\dfrac{x^{-2}y^2}{(xy)^{-3}}\right]^{-8}$.

94. $\{(a^{\frac{1}{2}}b^{\frac{5}{2}})^{\frac{1}{5}}\div(a^{-\frac{1}{2}}b)^{-2}\}^{\frac{1}{9}}$.

95. $\dfrac{a+b}{a^{\frac{1}{3}}-b^{\frac{1}{3}}}-\dfrac{a-b}{a^{\frac{1}{3}}+b^{\frac{1}{3}}}$.

96. $\dfrac{\left[-\dfrac{a^{-1}+b^{-1}}{a^{-1}-b^{-1}}\times(a^2-b^2)\right]^{\frac{1}{2}}}{\dfrac{b+a}{ab}}$.

Exercises on this page are from recent examination pape

97. Write the first five powers of $\sqrt{-1}$.

98. Prove that $(a^{\frac{1}{m}})^{\frac{1}{n}} = a^{\frac{1}{mn}}$; that $a^m \times a^{-n} = a^{m-n}$.

99. Which is the greater, $2^{\frac{7}{3}}$ or 5? Prove it.

100. Find the value of $\dfrac{1}{2+\sqrt{5}-\sqrt{2}}$ to two decimal plac

101. Extract the square root of

$$x(x-\sqrt{2})(x-\sqrt{8})(x-\sqrt{18})+4.$$

102. Find a factor that will rationalize $x^{\frac{2}{3}}+y^{\frac{1}{3}}$.

Simplify:

103. $16^{\frac{1}{4}}\cdot 2^{\frac{1}{2}}\cdot 32^{\frac{5}{6}}$.

104. $\sqrt{38-12\sqrt{10}}$.

105. $\dfrac{\sqrt{x}-\sqrt{x-2}}{\sqrt{x}+\sqrt{x-2}}$.

106. $\sqrt[3]{4^2}\cdot\sqrt[9]{8}\cdot 3\sqrt[3]{4}$.

107. $\sqrt[3]{-27}+(\sqrt{-1})^6+8^{-\frac{1}{3}}$.

108. $\sqrt{\dfrac{\sqrt{23}+\sqrt{7}}{\sqrt{23}-\sqrt{7}}}$.

109. $12^0+4^{\frac{1}{2}}-9^{-1}+\dfrac{1}{\sqrt{-64}}+27^{\frac{2}{3}}$.

110. $\dfrac{\sqrt{3}+\sqrt{2}}{2-\sqrt{3}}\div\dfrac{7+4\sqrt{3}}{\sqrt{3}-\sqrt{2}}$.

111. $\dfrac{\sqrt{3+\sqrt{5}}-\sqrt{3-\sqrt{5}}}{\sqrt{3+\sqrt{5}}+\sqrt{3-\sqrt{5}}}$.

112. $\dfrac{2\sqrt{15}+8}{5+\sqrt{15}}\div\dfrac{8\sqrt{3}-6\sqrt{5}}{5\sqrt{3}-3\sqrt{5}}$.

113. $\dfrac{b}{\sqrt{a}}\times\sqrt[3]{\dfrac{c}{a^{-1}}}\times\dfrac{\sqrt[4]{c^3}}{\sqrt{b}}\div\dfrac{a^{-\frac{1}{6}}}{b^{-\frac{1}{2}}}$.

114. $\dfrac{\dfrac{1}{1+x}-\dfrac{2}{x^2+3x+2}}{(x+2)^{-1}-(x+1)^{-1}(2+x)^{-1}}$.

QUADRATIC EQUATIONS

374. The equation $x - 2 = 0$ is of the *first* degree and has *one* root, $x = 2$. Similarly, $x - 3 = 0$ is of the first degree and has one root, $x = 3$. Consequently, the product of these two simple equations, which is

$$(x - 2)(x - 3) = 0, \text{ or } x^2 - 5x + 6 = 0,$$

is of the *second degree* and has *two* roots, 2 and 3.

375. An equation that, when simplified, contains the *square* of the unknown number, but no higher power, is called an equation of the *second degree*, or a **quadratic equation.**

It is evident, therefore, that quadratic equations may be of two kinds — those which contain only the second power of the unknown number, and those which contain both the second and first powers.

$x^2 = 15$ and $ax^2 + bx = c$ are quadratic equations.

PURE QUADRATIC EQUATIONS

376. An equation that contains only the second power of the unknown number is called a **pure quadratic.**

$ax^2 = b$ and $ax^2 - cx^2 = bc$ are pure quadratics.

Pure quadratics are called also **incomplete quadratics,** because they lack the first power of the unknown number.

377. Since pure quadratics contain only the second power of the unknown number, they may be reduced to the general form $ax^2 = b$, in which a represents the coefficient of x^2, and b the sum of the terms that do not involve x^2.

378. The equation $3x^2 = 300$ has two roots, for it may be reduced to the form $(x-10)(x+10)=0$, which is equivalent to the two simple equations,

$$x - 10 = 0 \text{ and } x + 10 = 0,$$

each of which has one root.

The roots, $+10$ and -10, are numerically equal but opposite in sign.

PRINCIPLE. — *Every pure quadratic equation has two roots, numerically equal but opposite in sign.*

It is proved in § 436 that *every* quadratic equation has two roots and only two roots.

EXERCISES

379. 1. Given $10x^2 = 99 - x^2$, to find the value of x.

SOLUTION

$$10x^2 = 99 - x^2.$$

Transposing, etc., $\quad 11x^2 = 99.$

Dividing by 11, $\quad x^2 = 9.$

Extracting the square root of each member, § 289,

$$x = \pm 3.$$

NOTE. — Strictly speaking, the last equation should be $\pm x = \pm 3$, which stands for the equations, $+x = +3$, $+x = -3$, and $-x = -3$, and $-x = +3$. But since the last two equations may be derived from the first two, by changing signs, the first two express *all* the values of x. For convenience, the two expressions, $x = +3$ and $x = -3$, are written $x = \pm 3$.

Consequently, in extracting the square roots of the members of an equation, it will be sufficient to write the double sign before the root of *one* member.

2. Find the roots of the equation $3x^2 = -15$.

SOLUTION

$$3x^2 = -15.$$

Dividing by 3, $\quad x^2 = -5.$

Extracting the square root, $\quad x = \pm\sqrt{-5}.$

VERIFICATION. — The given equation becomes $-15 = -15$ and is *therefore satisfied* when either $+\sqrt{-5}$ or $-\sqrt{-5}$ is substituted for x.

Solve for x, and verify each root:

3. $3x^2 - 5 = 22.$

4. $2x^2 + 3x^2 = 80.$

5. $4x^2 = \frac{1}{9}.$

6. $\frac{3}{4}x^2 - 5 = 22.$

7. $x^2 - b = 0.$

8. $6ax^2 - 54a^5 = 0.$

9. $7x^2 - 25 = 5x^2 + 73.$

10. $(x+4)^2 = 8x + 25.$

11. $(a-x)^2 = (3x+a)(x-a).$

12. $ax^2 = (a-b)(a^2-b^2) - bx^2.$

13. $a^2x^2 + 2ax^2 = (a^2-1)^2 - x^2.$

14. $(x+2)^2 - 4(x+2) = 4.$

15. $\dfrac{x-8}{6} = \dfrac{6}{x+8}.$

16. $\dfrac{1}{1-x} + \dfrac{1}{1+x} = \dfrac{8}{3}.$

17. $\dfrac{x}{12} + \dfrac{x^2-15}{5x} = \dfrac{x}{5}.$

18. $\dfrac{x+3}{x-3} + \dfrac{x-3}{x+3} = 4.$

19. $\dfrac{x-2}{x+1} + \dfrac{x+2}{x-1} = -1.$

20. $\dfrac{x-3}{x-2} + \dfrac{x+3}{x+2} = 1\frac{7}{8}.$

21. $\dfrac{a}{x} + \dfrac{x}{a} = \dfrac{ab}{x}.$

22. $\dfrac{x}{a+b} - \dfrac{a-b}{x} = 0.$

23. $\dfrac{x-2}{x+2} - \dfrac{x+2}{2-x} = \dfrac{40}{x^2-4}.$

24. $\dfrac{\sqrt{x^2+1} - \sqrt{x^2-1}}{\sqrt{x^2+1} + \sqrt{x^2-1}} = \dfrac{1}{2}.$

25. $\dfrac{x+a}{x-a} + \dfrac{x-a}{x+a} = \dfrac{2a}{1-a}.$

26. $\dfrac{x+a}{x+b} + \dfrac{x-a}{x-b} = \dfrac{a^2+b^2}{x^2-b^2}.$

27. $\sqrt{(x+3)(x-5)} = \sqrt{49-2x}.$

28. $\sqrt{25-6x} + \sqrt{25+6x} = 8.$

29. $\dfrac{x+7}{x^2-7x} - \dfrac{x-7}{x^2+7x} = \dfrac{7}{x^2-73}.$

30. $\dfrac{\sqrt{x+2a} - \sqrt{x-2a}}{\sqrt{x-2a} + \sqrt{x+2a}} = \dfrac{x}{2a}.$

31. $\dfrac{2}{x+\sqrt{2-x^2}} + \dfrac{2}{x-\sqrt{2-x^2}} = x.$

Problems

380. 1. What negative number is equal to its reciprocal?

2. If 25 is added to the square of a certain number, the sum is equal to the square of 13. What is the number?

3. What number is that whose square is equal to the difference of the squares of 25 and 20?

4. If a certain number is increased by 5 and also decreased by 5, the product of these results will be 75. What is the number?

5. How many rods of fence will inclose a square garden whose area is $2\frac{1}{2}$ acres?

6. The area of one side of a drawing board is 5 square feet. If it were 3 inches shorter and 3 inches wider, it would be square. Find its length and width.

7. The sum of two numbers is 10, and their product is 21. What are the numbers?

SUGGESTION. — Represent the numbers by $5 + x$ and $5 - x$.

8. The sum of two numbers is 16, and their product is 55. What are the numbers?

9. The sum of two numbers is 5, and their product is -14. What are the numbers?

10. Factor $a^2 + 17a + 60$ by the method suggested in the preceding problems.

SUGGESTION. — We need the two factors of 60 whose sum is 17. Represent them by $\frac{17}{2} + x$ and $\frac{17}{2} - x$. Then, $(\frac{17}{2} + x)(\frac{17}{2} - x) = 60$.

11. Separate $a^2 + 2a - 2$ into two factors.

12. Separate $x^2 - 2x - 1$ into two factors.

13. Divide 24 into two parts whose product is 143.

14. The sum of the squares of two numbers is 394, and the *difference* of their squares is 56. What are the numbers?

The length of a 10-acre field is 4 times its width. What s dimensions?

At 75 cents per square yard, enough linoleum was pur- d for $36 to cover a rectangular floor whose length 3 times its breadth. What were the dimensions of the ?

The lock of the St. Mary's Canal, Michigan, is 8 times ng as it is wide; the surface of the water it contains is 0 square feet in area. What are the dimensions of the

A man has two square fields that together contain $51\frac{1}{4}$ If the side of one is as much longer than 50 rods as f the other is shorter than 50 rods, what are the dimen- of each field?

A man had a rectangular field whose width was $\frac{3}{8}$ of its h. He built a fence across it so that one of the two parts formed was a square. If the square field contained res, what were the dimensions of the original field?

A shipment of railroad ties measuring 400,000 board ontained as many car loads as there were board feet in If each car held 250 ties, find the total number of ties he number of board feet in one tie.

Formulæ

Solve the following formulæ from physics:

$s = \frac{1}{2} gt^2$, for t.

$E = \frac{1}{2} Mv^2$, for v.

$P = I^2R$, for I.

4. $F = \frac{mv^2}{R}$, for v.

5. $G = \frac{mm'}{d^2}$, for d.

When $g = 32.16$, formula 1 gives the number of feet (s) gh which a body will fall in t seconds, starting from How long will it take a brick to fall to the sidewalk *the top of a building* 100.5 feet high?

7. To lighten a balloon at the height of 2500 feet, a bag of sand was let fall. Find the time, to the nearest tenth of a second, required for it to reach the earth.

Solve the following geometrical formulæ:

8. $c^2 = a^2 + b^2$, for b.

9. $4m^2 = 2(a^2 + b^2) - c^2$, for m.

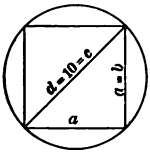
10. $A = .7854\,d^2$, for d.

11. $V = \frac{1}{3}\pi r^2 h$, for r.

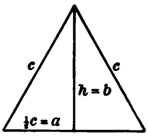
12. Using formula 8, find the hypotenuse (c) of a right triangle whose other two sides are $a = 8$ and $b = 6$.

13. Solve formula 8 for b, and find the side (b) of a right triangle whose hypotenuse (c) is 5 and whose side (a) is 3.

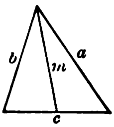
14. From formula 8 and the accompanying figure find, to the nearest tenth, the side (a) of a square inscribed in a circle whose diameter (d) is 10.



15. Find, to the nearest tenth of an inch, the dimensions of the largest square timber that can be cut from a log 12 feet long and 18 inches in diameter.



16. Using formula 8 and the accompanying figure, deduce a formula for the altitude (h) of an equilateral triangle in terms of its side (c).



17. From formula 9, find the length of the median (m) to the side (c) of the triangle in the accompanying figure, if $a = 11$, $b = 8$, and $c = 9$.

18. Substituting in formula 10, find, to the nearest tenth of a foot, the diameter (d) of a circle whose area (A) is 1000 square feet.

19. Using formula 11, find, to the nearest centimeter, the radius (r) of the base of a conical vessel 20 centimeters high ($h = 20$) that will hold a liter of water ($V = 1$ liter $= 1000$ *cu. cm.*; $\pi = 3.1416$).

AFFECTED QUADRATIC EQUATIONS

2. A quadratic equation that contains both the second and irst powers of one unknown number is called an **affected** **ratic.**

$x = 10$, $4x^2 - x = 3$, and $ax^2 + bx + c = 0$ are affected quadratics.

fected quadratics are sometimes called **complete quadratics.**

3. Since affected quadratic equations contain both the d and the first powers of the unknown number, they may ys be reduced to the general form of $ax^2 + bx + c = 0$, in h a, b, and c may represent any numbers whatever, and unknown number.

e term c is called the **absolute term.**

4. To solve affected quadratics by factoring.

duce the equation to the form $ax^2 + bx + c = 0$, factor the member, and equate each factor to zero, as in § 172, thus ning two simple equations together equivalent to the given ratic, subject to the exceptions given in § 230 as to alence.

us, $3x^2 = 10x - 3.$

nsposing, $3x^2 - 10x + 3 = 0.$

toring, $(x - 3)(3x - 1) = 0.$

$\therefore x - 3 = 0$ or $3x - 1 = 0$;

e, $x = 3$ or $\frac{1}{3}$.

EXERCISES

Solve by factoring, and verify results:

$x^2 - 5x + 6 = 0.$

$x^2 + 10x + 21 = 0.$

$x^2 + 12x - 28 = 0.$

$x^2 - 20x + 51 = 0.$

$x^2 - 5x = 24.$

$x^2 - 1 = 3(x + 1).$

7. $2x^2 - 7x + 3 = 0.$

8. $2z^2 - z - 3 = 0.$

9. $3v^2 - 2v - 8 = 0.$

10. $10r^2 - 27r + 5 = 0.$

11. $6(s^2 + 1) = 13s.$

12. $2x^2 + 7x = 4.$

386. First method of completing the square.

Since $$(x+a)^2 = x^2 + 2\,ax + a^2,$$

the general form of the perfect square of a binomial is

$$x^2 + 2\,ax + a^2.$$

Consequently, an expression like $x^2 + 2\,ax$ may be made a perfect square by adding the term a^2, which it will be observed is *the square of half the coefficient of* x.

Thus, to solve $$x^2 + 6\,x = -5$$

by the method of extracting the square root of both members (the method used in solving pure quadratics), we must **complete the square** in the first member.

The number to be added is the square of half the coefficient of x; that is, $(\frac{6}{2})^2$, or 9. The same number must be added to the second member to preserve the equality.

Therefore, Ax. 1, $$x^2 + 6\,x + 9 = -5 + 9;$$

that is, $$x^2 + 6\,x + 9 = 4.$$

Extracting the square root, § 289, $$x + 3 = \pm 2;$$

whence, $$x = -3 + 2 \text{ or } -3 - 2.$$

$$\therefore x = -1 \text{ or } -5.$$

EXERCISES

387. **1.** Solve the equation $x^2 - 5\,x - 14 = 0$.

SOLUTION

$$x^2 - 5\,x - 14 = 0.$$

Transposing, $$x^2 - 5\,x = 14.$$

Completing the square, $$x^2 - 5\,x + \tfrac{25}{4} = 14 + \tfrac{25}{4} = \tfrac{81}{4}.$$

Extracting the square root, $$x - \tfrac{5}{2} = \pm \tfrac{9}{2};$$

whence, $$x = \tfrac{5}{2} + \tfrac{9}{2} \text{ or } \tfrac{5}{2} - \tfrac{9}{2}.$$

$$\therefore x = 7 \text{ or } -2.$$

VERIFICATION. — Either 7 or -2 substituted for x in the given equation *reduces it to* $0 = 0$, *an identity; that is, the given equation is satisfied by these values of* x.

Solve the equation $4x^2 + 4x + 6 = 0$.

Solution

$$4x^2 + 4x + 6 = 0.$$

ransposing, $\qquad 4x^2 + 4x = -6.$

ividing by 4, $\qquad x^2 + x = -\frac{3}{2}.$

ompleting the square, $\qquad x^2 + x + \frac{1}{4} = -\frac{3}{2} + \frac{1}{4} = -\frac{5}{4}.$

Extracting the square root, $\qquad x + \frac{1}{2} = \pm \frac{1}{2}\sqrt{-5}.$

$$\therefore x = -\frac{1}{2} + \frac{1}{2}\sqrt{-5} \text{ or } -\frac{1}{2} - \frac{1}{2}\sqrt{-5},$$

ich would usually be written, $\qquad x = \frac{1}{2}(-1 \pm \sqrt{-5}).$

teps in the solution of an affected quadratic equation by first method of completing the square are:

1. *Transpose so that the terms containing x^2 and x are in one mber and the known terms in the other.*
2. *Make the coefficient of x^2 positive unity by dividing both mbers by the coefficient of x^2.*
3. *Complete the square by adding to each member the square half the coefficient of x.*
4. *Extract the square root of both members.*
5. *Solve the two simple equations thus obtained.*

Solve, and verify all results:

3. $x^2 - 2x = 143.$
4. $x^2 + 2x = 168.$
5. $x^2 - 4x = 117.$
6. $x^2 - 6x = 160.$
7. $8x = x^2 - 180.$
8. $x^2 + 2x = 120.$
9. $x^2 + 22x = -120.$
10. $x^2 = 28x - 187.$
11. $x^2 - 12x = 189.$
12. $y^2 = 10 - 3y.$
13. $z^2 - 180 = 3z.$
14. $v^2 + 15v = 54.$
15. $v^2 + 21v = -54.$
16. $n(n-1) = 930.$
17. $r^2 + 27r + 140 = 0.$
18. $l^2 - 11l + 28 = 0.$
19. $5x^2 - 3x - 2 = 0.$
20. $6x^2 - 5x - 6 = 0.$

21. $.2x^2 + .9x = 3.5.$

22. $.03x^2 - .07x = .1.$

23. $2x^2 - \frac{19}{2}x = \frac{5}{2}.$

24. $\dfrac{1}{x+1} + \dfrac{3}{x-1} = \dfrac{10}{3}.$

25. $\dfrac{x^2}{x-2} - \dfrac{3x-5}{2} = \dfrac{x+2}{5}.$

388. Other methods of completing the square.

To apply the first method of completing the square, the coefficient of x^2 must be $+1$ or be made $+1$.

Other methods of completing the square are based on making the coefficient of x^2 a perfect square, if it is not already one, by multiplying or dividing both members of the equation by some number.

Thus, given $\qquad 3x^2 + 10x = -3.$

Multiplying by 3, $\qquad 9x^2 + 30x = -9.$

When the square of the first member is completed, $30x$ will be *twice* the product of the square roots of the terms that are squares. Hence, the square root of the term to be added is $15x \div \sqrt{9x^2}$, or 5; and the number to be added is 5^2, or 25.

Completing the square,

$$9x^2 + 30x + 25 = -9 + 25 = 16.$$

When the coefficient of x^2 has been made a perfect square, the number to be added to complete the trinomial square is obtained by dividing half the term containing x by the square root of the term containing x^2, and squaring the quotient.

EXERCISES

389. 1. Solve the equation $8x^2 - 10x = 3$.

SOLUTION

$$8x^2 - 10x = 3.$$

Multiplying by 2, $\qquad 16x^2 - 20x = 6.$

Completing the square, $\qquad 16x^2 - 20x + \frac{25}{4} = 6 + \frac{25}{4} = \frac{49}{4}.$

Extracting the square root, $\qquad 4x - \frac{5}{2} = \pm \frac{7}{2}.$

$$4x = \frac{5}{2} \pm \frac{7}{2} = 6 \text{ or } -1.$$

$$\therefore x = \frac{3}{2} \text{ or } -\frac{1}{4}.$$

Solve, and verify:

2. $2x^2 - 5x = 42$.

3. $6x^2 - 5x + 1 = 0$.

4. $4x^2 - 12x = 27$.

5. $18x^2 + 6x = 4$.

6. $2x^2 - 11x + 12 = 0$.

7. $3x^2 + 4x = 95$.

8. $7v^2 + 2v = 32$.

9. $8x^2 - 18x = 5$.

10. $6m^2 + 5m = 4$.

11. $5n^2 - 14n = -8$.

12. Solve the general quadratic equation $ax^2 + bx + c = 0$.

SOLUTION

$$ax^2 + bx + c = 0. \quad (1)$$

Transposing c, $$ax^2 + bx = -c. \quad (2)$$

Multiplying by a, $$a^2x^2 + abx = -ac. \quad (3)$$

Completing the square, $$a^2x^2 + abx + \frac{b^2}{4} = \frac{b^2}{4} - ac. \quad (4)$$

Multiplying by 4, $$4a^2x^2 + 4abx + b^2 = b^2 - 4ac. \quad (5)$$

Extracting the square root, $$2ax + b = \pm\sqrt{b^2 - 4ac}. \quad (6)$$

$$\therefore x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \quad (7)$$

It is evident that (5) can be obtained by multiplying (2) by $4a$ and adding b^2 to both members. Hence, when a quadratic has the general form of (1), if the absolute term is transposed to the second member, as in (2), the square may be completed and **fractions avoided** by

Multiplying by 4 times the coefficient of x^2 and adding to each member the square of the coefficient of x in the given equation.

This is called the **Hindoo method** of completing the square.

Solve by the Hindoo method, and verify results:

13. $2x^2 + 3x = 27$.

14. $2x^2 + 5x = 7$.

15. $2x^2 + 7x = -6$.

16. $3x^2 - 7x = -2$.

17. $4x^2 - 17x = -4$.

18. $4x^2 - x - 3 = 0$.

19. $5x^2 - 2x - 16 = 0$.

20. $3x^2 + 7x - 110 = 0$.

21. $2x^2 - 5x - 150 = 0$.

22. $3x^2 + x - 200 = 0$.

23. $5x^2 - 7x = -2.$

24. $6x^2 + 5x = -1.$

25. $15x^2 - 7x - 2 = 0.$

26. $7x^2 - 20x - 32 = 0.$

390. To solve quadratics by a formula.

The general quadratic

$$ax^2 + bx + c = 0$$

has been solved in exercise 12, § 389. Its roots are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Since (1) represents *any* quadratic equation, the student now prepared to solve any quadratic equation whatever, tl contains one unknown number.

The roots of any quadratic equation, then, may be obtained reducing it to the general form and employing (2) as a formı

EXERCISES

391. 1. Solve the equation $6x^2 = x + 15$.

SOLUTION. — Writing the equation in the general form

$$6x^2 - x - 15 = 0,$$

we find that $a = 6$, $b = -1$, and $c = -15$.

$$\therefore \text{ by (2), § 390, } x = \frac{1 \pm \sqrt{(-1)^2 - 4 \times 6(-15)}}{2 \times 6}$$

$$= \frac{1 \pm 19}{12} = \frac{5}{3} \text{ or } -\frac{3}{2}.$$

Solve by the above formula, and verify results:

2. $2x^2 + 5x + 2 = 0.$

3. $3x^2 + 11x + 6 = 0.$

4. $6x^2 + 2 = 7x.$

5. $4x^2 + 4x = 15.$

6. $2x^2 = 9 - 3x.$

7. $x(2x + 3) = -1.$

8. $13x = 3x^2 - 10.$

9. $7x^2 + 9x = 10.$

10. $1 - 3x = 2x^2.$

11. $4 = x(3x + 2).$

12. $x^2 - 5x = -3.$

13. $3x^2 - 6x = -2.$

14. $4x^2 - 3x - 2 = 0.$

15. $x^2 + 10 = 6x.$

16. $x^2 = -4(x + 3).$

17. $4(2x - 5) = x^2.$

392. Miscellaneous equations to be solved by any method.

EXERCISES

1. Solve the equation $3x^2 + 2x = 0$.

REMARK. — Dividing by x removes the root $x = 0$ and reduces the equation to the simple equation $3x + 2 = 0$, whose root is $x = -\frac{2}{3}$.

If the given equation is solved by quadratic methods, the roots are found to be the same, namely, 0 and $-\frac{2}{3}$; consequently, it is important to account for roots that may be *removed* (§ 230) by dividing by an expression that involves the unknown number. The root removed is the root of the equation formed by equating the divisor to 0.

2. Solve the equation $\dfrac{4x}{x-1} - \dfrac{x^2+3x}{x^2} = 4$.

SOLUTION. $$\frac{4x}{x-1} - \frac{x^2+3x}{x^2} = 4.$$

First reducing the second fraction to its lowest terms, then multiplying both members by the L.C.D., $x(x-1)$, simplifying, etc., we have

$$x^2 - 2x - 3 = 0.$$

Factoring, $$(x-3)(x+1) = 0.$$

$$\therefore x = 3 \text{ or } -1.$$

VERIFICATION. — When $x = 3$, each member $= 4$; when $x = -1$, each member $= 4$; that is, both $x = 3$ and $x = -1$ are found to be roots of the given equation.

NOTE. — If the second fraction is not reduced to lowest terms before clearing the equation of fractions, the multiplier is $x^2(x-1)$ instead of $x(x-1)$, and the root $x = 0$ so introduced must be rejected.

In general, *no root is introduced by clearing an equation of fractions, provided that: fractions having a common denominator are combined; each fraction is expressed in its lowest terms; and both members are then multiplied by the lowest common denominator.*

General Directions. — 1. *Reduce the equation to the general form* $ax^2 + bx + c = 0$.

2. *If the factors are readily seen, solve by factoring.*

3. *If the factors are not readily seen, solve by completing the square or by formula.*

4. *Verify all results, reject roots introduced in the process of reducing the equation to the general form, and account for roots that have been removed.*

Solve according to the general directions just given:

3. $x^2-6x+5=0.$

4. $2x^2-5x=0.$

5. $7x^2+2x=32.$

6. $x^2=3x+10.$

7. $x^2-30=13x.$

8. $x^2-12x=28.$

9. $x^2-12x=0.$

10. $18x^2+6x=0.$

11. $4x^2-12x=0.$

12. $x^2-4.3x=27.3.$

13. $x^2+.25x=.15.$

14. $x+\frac{1}{x}-\frac{5}{2}=0.$

15. $\frac{x^2}{9}+\frac{x^2-2x}{3x-6}=\frac{35}{4}.$

16. $\frac{x}{9(x-1)}=\frac{x-2}{6}.$

17. $\frac{4}{x^2-2x+1}=\frac{1}{4}.$

18. $\frac{x^2}{4}-\frac{2x}{3}=28.$

19. $\frac{9x}{2x^2+x}+\frac{3}{x-3}=4.$

20. $\frac{1+x}{x-3}-\frac{x-1}{x-2}=\frac{4}{5}.$

21. $\frac{x}{x-5}-\frac{x-5}{x}=\frac{3}{2}.$

22. $\frac{x+7}{x+5}+\frac{x+12}{x+6}=7.$

23. $\frac{x+4}{x-2}+3=\frac{(x+3)^2}{x^2-9}.$

24. $\frac{x^2}{x-2}=\frac{4}{x-2}+5.$

25. $\frac{x}{x+2}+\frac{1}{2}=\frac{x+2}{2x}.$

26. $\frac{5x}{x+7}+\frac{x+6}{x+3}=3.$

27. $\frac{x+2}{x-7}-\frac{x+5}{x-5}=1.$

28. $\frac{x-3}{x+4}+\frac{x+2}{x-2}=\frac{23}{10}.$

29. $\frac{2x+1}{1-2x}-\frac{5}{7}=\frac{x-8}{2}.$

30. $\frac{2x-3}{x^2-3x}=2-\frac{3}{x^2-3x}.$

Find roots to the nearest thousandth:

31. $x^2-4x-1=0.$

32. $v^2+6v+7=0.$

33. $u^2+5u+5.5=0.$

34. $t^2-12t+16.5=0.$

Literal Equations

393. The methods of solution for literal quadratic equations are the same as for numerical quadratics. The method by factoring (§ 384) is recommended when the factors can be seen readily. If it is necessary to complete the square, the first method (§ 386) is usually more advantageous, provided the coefficient of x^2 is $+1$, otherwise the Hindoo method (§ 389) is better, because by its use fractions are avoided. Results may be tested by substituting simple numerical values for the literal known numbers.

EXERCISES

394. Solve for x by the method best adapted:

1. $x^2 - ax = ab - bx.$
2. $x^2 + ax = ac + cx.$
3. $x^2 = (m - n)x + mn.$
4. $5x - 2ax = x^2 - 10a.$
5. $x^2 + 3bx = 5cx + 15bc.$
6. $6x^2 + 3ax = 2bx + ab.$
7. $acx^2 - bcx - bd + adx = 0.$
8. $x^2 + 4mx + 3nx + 12mn = 0.$
9. $x^2 = 4ax - 2a^2.$
10. $x^2 - ax + a^2 = 0.$
11. $4ax - x^2 = 3a^2.$
12. $5ax + 6a^2 = 6x^2.$
13. $21b^2 - 4bx = x^2.$
14. $\dfrac{7m^2}{12} - mx = \dfrac{x^2}{3}.$
15. $\dfrac{x^2}{3b} = \dfrac{5x}{4} + \dfrac{b}{3}.$
16. $\dfrac{x}{x-1} - \dfrac{x}{x+1} = m.$
17. $x + \dfrac{a^2}{x} = \dfrac{a^2}{b} + b.$
18. $2x - \dfrac{3x^2}{a} = a - 2x.$
19. $\dfrac{1}{ax+4} = 1 - \dfrac{ax-4}{16}.$
20. $x^2 + \dfrac{a}{b}x = \dfrac{a+b}{b}.$
21. $x^2 + 2 = \left(\dfrac{2a^2+1}{a}\right)x.$
22. $x^2 - \dfrac{2x}{ab} = \dfrac{4(ab-1)}{ab}.$

23. $x^2 - 2(a - b)x = 4\,ab.$

24. $x^2 - 2\,x(m - n) = 2\,mn.$

25. $x^2 + 2(a + 8)x = -32\,a.$

26. $x^2 + x + bx + b = a(x + 1).$

27. $a(2\,x - 1) + 2\,bx - b = x(2\,x - 1).$

28. $x^2 + 4(a - 1)x = 8\,a - 4\,a^2.$

29. $\dfrac{1}{a + b + x} = \dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{x}.$

30. $\dfrac{x^2 + 1}{n^2x - 2\,n} - \dfrac{1}{2 - nx} = \dfrac{x}{n}.$

31. $\dfrac{2\,a + x}{2\,a - x} + \dfrac{a - 2\,x}{a + 2\,x} = \dfrac{8}{3}.$

32. $\dfrac{1}{a - x} - \dfrac{1}{a + x} = \dfrac{3 + x^2}{a^2 - x^2}.$

33. $a(x - 2\,a + b) + a(x + a - b) = x^2 - (a - b)^2.$

34. $\dfrac{x^2}{a + b} - \left(1 + \dfrac{1}{ab}\right)x + \dfrac{1}{a} + \dfrac{1}{b} = 0.$

35. $\dfrac{x^2 + 1}{x} - \dfrac{a + b}{c} = \dfrac{c}{a + b}.$

36. $\dfrac{2\,x - a}{b} + 3 = \dfrac{4\,a}{2\,x - b}.$

37. $\dfrac{bx}{a - x} + b = \dfrac{a(x + 2\,b)}{a + b}.$

38. $\dfrac{3\,x + b}{x + b} = \dfrac{b}{2\,x - a} + \dfrac{1}{1 + \dfrac{x - a}{a + b}}.$

39. $\dfrac{x^4}{a^2} + \left(x + \dfrac{ab}{x}\right)^2 - \left(\dfrac{x^2}{a} + \dfrac{ab}{x}\right)^2 = ax.$

Radical Equations

395. In §§ 364, 365, the student learned how to free radical equations of radicals, the cases treated there being such as lead to simple equations. The radical equations in this chapter lead to quadratic equations, but the methods of freeing them of radicals are the same as in the cases already discussed.

396. The principles of § 230 in regard to the *equivalence* of equations have been illustrated in § 392, exercise 1, showing the removal of a root by dividing by an unknown expression, and exercise 2, showing the *introduction* of a root by clearing of fractions unless certain precautions are taken. In the discussion of § 366 it was shown that the processes of *rationalization* and *involution*, used in freeing radical equations of radicals, are likely to introduce roots according to the convention adopted there as to how roots shall be verified.

Hence, it is important in the solution of equations that roots be tested not only to determine the accuracy of the work, but to discover whether the solutions obtained are really roots of the given equation, and also to examine the processes employed in reducing equations to see whether any roots have been removed.

EXERCISES

397. 1. Solve the equation $2\sqrt{x} - x = x - 8\sqrt{x}$.

SOLUTION

$$2\sqrt{x} - x = x - 8\sqrt{x}.$$

Dividing by $\sqrt{x}$, $\quad 2 - \sqrt{x} = \sqrt{x} - 8.$

Transposing, etc., $\quad \sqrt{x} = 5.$

Squaring, $\quad x = 25.$

VERIFICATION. — When $x = 25$,

$$\text{1st member} = 2\sqrt{25} - 25 = 10 - 25 = -15;$$

$$\text{2d member} = 25 - 8\sqrt{25} = 25 - 40 = -15.$$

Hence, $x = 25$ is a root of the equation; $x = 0$, the root of the equation $\sqrt{x} = 0$, also is a root of the given equation, removed by dividing both members by $\sqrt{x}$.

2. Solve and verify $\sqrt{x+1}+\sqrt{x-2}-\sqrt{2x-5}=0$.

Solution

$$\sqrt{x+1}+\sqrt{x-2}-\sqrt{2x-5}=0.$$

Transposing, $\quad \sqrt{x+1}+\sqrt{x-2}=\sqrt{2x-5}.$

Squaring, $\quad x+1+2\sqrt{x^2-x-2}+x-2=2x-5.$

Simplifying, $\quad \sqrt{x^2-x-2}=-2.$

Squaring, $\quad x^2-x-2=4.$

Solving, $\quad x=-2 \text{ or } 3.$

Verification. — Substituting -2 for x in the given equation,

$$\sqrt{-1}+\sqrt{-4}-\sqrt{-9}=0;$$

that is, $\quad \sqrt{-1}+2\sqrt{-1}-3\sqrt{-1}=0.$

Therefore, -2 is a root of the given equation.

Substituting 3 for x in the given equation,

$$\sqrt{4}+\sqrt{1}-\sqrt{1}=0,$$

which is not true according to the convention adopted in the discussion in § 366. Hence, 3 is not to be regarded as a root of the given equation.

Note. — The equation could be verified for $x=3$ if the negative square root of 1 were taken in the second term and the positive square root in the third, thus:

$$\sqrt{4}+\sqrt{1}-\sqrt{1}=2+(-1)-(+1)=0.$$

This is an improper method of verification, however, for it has been agreed previously that the square root sign shall denote only the *positive* square root.

Solve and verify, rejecting roots that do not satisfy the given equation, and accounting for roots that otherwise might be lost:

3. $8\sqrt{x}-8x=\frac{3}{2}$.

4. $3x+\sqrt{x}=5\sqrt{4x}$.

5. $x-1+\sqrt{x+5}=0$.

6. $x-5-\sqrt{x-3}=0$.

7. $\sqrt{4x+17}+\sqrt{x+1}-4=0$.

8. $1+\sqrt{(3-5x)^2+16}=2(3-x)$.

9. $\sqrt{1+x\sqrt{x^2+12}}=1+x.$

10. $\sqrt{x-1}+\sqrt{2x-1}-\sqrt{5x}=0.$

11. $\sqrt{2x-7}-\sqrt{2x}+\sqrt{x-7}=0.$

12. $\sqrt{x+3}+\sqrt{4x+1}-\sqrt{10x+4}=0.$

13. $\sqrt{a+x}-\sqrt{a-x}=\sqrt{2x}.$

14. $\sqrt{x-a}+\sqrt{b-x}=\sqrt{b-a}.$

15. $\sqrt{x^2-b^2}=\sqrt{x+b}\sqrt{a+b}.$

16. $\sqrt{2x+\sqrt{10x+1}}=\sqrt{2x}+1.$

17. $\sqrt{6+x}+\sqrt{x}-\sqrt{10-4x}=0.$

18. $\sqrt{4x-3}-\sqrt{2x+2}=\sqrt{x-6}.$

19. $\sqrt{2x+3}-\sqrt{x+1}=\sqrt{5x-14}.$

20. $\sqrt{3x-5}+\sqrt{x-9}=\sqrt{4x-4}.$

21. $\sqrt{x^2+8}-\dfrac{6}{\sqrt{x^2+8}}=x.$

22. $x+\sqrt{x^2+m^2}=\dfrac{2m^2}{\sqrt{x^2+m^2}}.$

23. $x+\sqrt{x^2-a^2}=\dfrac{a^2}{\sqrt{x^2-a^2}}.$

24. $\dfrac{2x+\sqrt{4x^2-1}}{2x-\sqrt{4x^2-1}}=4.$

25. $\sqrt{\dfrac{x-a}{x+a}}+\sqrt{\dfrac{x+a}{x-a}}=a^2.$

26. $\sqrt{a-x}+\sqrt{b-x}=\sqrt{a+b-2x}.$

27. $\sqrt{x+a^2}-\sqrt{x-2a^2}=\sqrt{2x-5a^2}.$

28. $\sqrt{mn-x}-\sqrt{x}\sqrt{mn-1}=\sqrt{mn}\sqrt{1-x}.$

Problems

398. 1. The sum of two numbers is 8, and their product is 15. Find the numbers.

SOLUTION

Let $x =$ one number.

Then, $8 - x =$ the other.

Since their product is 15, $(8 - x)x = 15$.

Solving, $x = 3$ or 5,

and $8 - x = 5$ or 3.

Therefore, the numbers are 3 and 5.

2. Divide 20 into two parts whose product is 96.

3. Divide 14 into two parts whose product is 45.

4. Find two consecutive integers the sum of whose squares is 61.

5. A rectangular garden is 12 rods longer than it is wide and it contains 1 acre. What are its dimensions?

6. A plumber received \$24 for some work. The number of hours that he worked was 20 less than the number of cents per hour that he earned. Find his hourly wage.

7. The 1860 bunches of asparagus from an acre of land were sold in boxes each holding 1 less than $\frac{1}{2}$ as many bunches as there were boxes. Find the number of bunches in a box.

8. Some boys laid out basket-ball grounds 30 feet greater in length than in width, but in order to bring the area down to the prescribed limit of 3500 square feet, they reduced the length 10 feet. How much too large had they laid out the grounds?

9. The volume of a standard size of concrete building blocks was 2880 cubic inches. The smallest dimension was *9 inches,* and the greatest was 13 inches more than the sum of *the other two.* What were the three dimensions?

rty hired a coach for $12. In consequence of the ıree of them to pay, each of the others had to pay re. How many persons were in the party?

Solution

$x =$ the number of persons.

$x - 3 =$ the number that paid,

$\frac{12}{x} =$ the number of dollars each should have paid,

$\frac{12}{x-3} =$ the number of dollars each paid.

$$\frac{12}{x-3} - \frac{1}{5} = \frac{12}{x}.$$

$$x = 15 \text{ or } -12.$$

l value of x is evidently inadmissible, since there could not number of persons. Hence, the number of persons in the

ıb had a dinner that cost $60. If there had been ıore, the share of each would have been $1 less. persons were there in the club?

rty of young people agreed to pay $8 for a sleigh were obliged to be absent, the cost for each of the ¢ greater. How many went on the ride?

ɔ of dairy butter weighed 20 pounds less than a tub · butter, and 360 pounds of dairy butter required 3 han the same amount of creamery butter. What ıtter was there in a tub of each kind?

ɔving picture film 150 feet long is made up of a ber of individual pictures. If these pictures were ɔh longer there would be 600 less for the same m. How long is each separate picture?

coats of paint applied to the sides of a barn having 95 square yards required 69 pounds of paint. One ·ed $1\frac{1}{2}$ square yards more for the second coat than t. What area did 1 pound of paint cover for each

16. A purchase of 80 four-inch spikes weighed 3 pounds less than one of 80 five-inch spikes. If 1 pound of the former contained 6 spikes more than 1 pound of the latter, how many of each kind weighed 1 pound?

17. Mr. Field paid $8.00 for one mile of No. 9 steel wire and $2.88 for one mile of No. 14 wire. The No. 9 wire weighed 224 pounds more, and cost $\frac{1}{2}$¢ per pound less, than the No. 14 wire. Find the cost of each per pound.

18. A train started 16 minutes late, but finished its run of 120 miles on time by going 5 miles per hour faster than usual. What was the usual rate per hour?

19. To run around a track 1320 feet in circumference took one man 5 seconds less time than it took another who ran 2 feet per second slower. How long did it take each man?

20. Two automobiles went a distance of 60 miles, one making 6 miles per hour faster time than the other and completing the journey $\frac{5}{6}$ of an hour sooner. How long was each on the way?

21. A cistern can be filled by two pipes in 24 minutes. If it takes the smaller pipe 20 minutes longer to fill the cistern than the larger pipe, in what time can the cistern be filled by each pipe?

SOLUTION

Let $x =$ the number of minutes required by the larger pipe.

Then, $x + 20 =$ the number of minutes required by the smaller.

Since $\frac{1}{x} =$ the part which the larger pipe fills in one minute,

$\frac{1}{x+20} =$ the part which the smaller pipe fills in one minute,

and $\frac{1}{24} =$ the part which both pipes fill in one minute,

then, $$\frac{1}{x} + \frac{1}{x+20} = \frac{1}{24}.$$

Solving, $$x = 40 \text{ or } -12.$$

Hence, the larger pipe can fill the cistern in 40 minutes, and the smaller pipe in 60 minutes.

22. A city reservoir can be filled by two of its pumps in ꞉days. The larger pump alone would take $1\frac{3}{4}$ days less time han the smaller. In what time can each fill the reservoir?

23. A company owned two plants that together made 25,200 oncrete building blocks in 12 days. Working alone, one plant ould have required 7 days more time than the other. What as the daily capacity of each plant?

24. The number of strawberry baskets made by a machine as 12 more per minute than the number of peach baskets ade by another machine. One day the former machine started 5 minutes after the latter, but each finished 2400 baskets at e same instant. Find the rate of each per minute.

25. Find two consecutive integers the sum of whose reciprocals is $\frac{9}{20}$.

26. Find the price of eggs, when 2 less for 30 cents raises e price 2 cents per dozen.

27. A merchant sold a hunting coat for $11, and gained a er cent equal to the number of dollars the coat cost him. hat was his per cent of gain?

28. By receiving two successive discounts, a dealer bought lverware listed at $20 for $9. What were the discounts in er cent, if the first was 5 times the second?

29. In gathering $187\frac{1}{2}$ bushels of tomatoes, the pickers used skets that held $\frac{1}{8}$ of a bushel less than the crates in which e tomatoes were shipped. The number of basketfuls picked as 50 more than the number of crates filled. Find the capaty of a basket; of a crate.

30. Each page of a book of 400 pages was 10 inches by 6 ches. The publishers decided to save 1550 square inches of per on each book in future editions by cutting down the marn equally on every side. By what width was the margin duced?

SUGGESTION. — The number of leaves in each book = $\frac{1}{2}$ the number of ges.

Formulæ

399. **1.** In any right-angled triangle (Fig. 1), $c^2 = a^2 + b^2$. Find all the sides when $a = c - 2$ and $b = c - 4$.

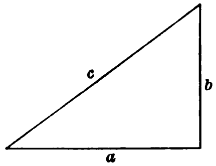


FIG. 1.

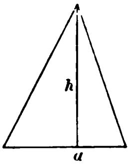


FIG. 2.

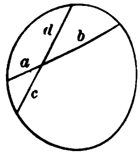


FIG. 3.

2. The area (A) of a triangle (Fig. 2) is expressed by the formula $A = \frac{1}{2}ah$. If the altitude (h) of a triangle is 2 inches greater than the base (a) and the area is 60 square inches, what is the length of the base?

3. If two chords intersect in a circle, as shown in Fig. 3, $a \cdot b$ is always equal to $c \cdot d$. Compute a and b when $c = 4$, $d = 6$, and $b = a + 5$.

4. The formula $h = a + vt - 16t^2$ gives, approximately, the height (h) of a body at the end of t seconds, if it is thrown vertically *upward*, starting with a velocity of v feet per second from a position a feet high.

Solve for t, and find how long it will take a skyrocket to reach a height of 796 feet, if it starts from a platform 12 feet high with an initial velocity of 224 feet per second.

5. How long will it take a bullet to reach a height of 25,600 feet, if it is fired vertically upward from the level of the ground with an initial velocity of 1280 feet per second?

6. When a body is thrown vertically *downward*, an approximate formula for its height is $h = a - vt - 16t^2$, in which h, a, v, and t stand for the same elements as in exercise 4.

Solve for t, and find when a ball thrown vertically downward from the Eiffel tower, height 984 feet, with an initial velocity of 24 feet per second, will be 368 feet above ground.

Find, to the nearest second, when it will reach the ground.

EQUATIONS IN THE QUADRATIC FORM

0. An equation that contains but two powers of an un-
vn number or expression, the exponent of one power being
e that of the other, as $ax^{2n} + bx^n + c = 0$, in which n repre-
s any number, is in the **quadratic form.**

EXERCISES

1. 1. Given $x^4 + 6x^2 - 40 = 0$, to find the values of x.

SOLUTION

$$x^4 + 6x^2 - 40 = 0.$$

ctoring, $(x^2 - 4)(x^2 + 10) = 0.$

$$\therefore x^2 - 4 = 0 \text{ or } x^2 + 10 = 0,$$

$$x = \pm 2 \text{ or } \pm\sqrt{-10}.$$

Given $x^{\frac{1}{2}} - x^{\frac{1}{4}} = 6$, to find the values of x.

FIRST SOLUTION

$$x^{\frac{1}{2}} - x^{\frac{1}{4}} = 6.$$

mpleting the square, $x^{\frac{1}{2}} - x^{\frac{1}{4}} + (\frac{1}{2})^2 = \frac{25}{4}.$

tracting the square root, $x^{\frac{1}{4}} - \frac{1}{2} = \pm \frac{5}{2}.$

$$\therefore x^{\frac{1}{4}} = 3 \text{ or } -2.$$

aising to the fourth power, $x = 81 \text{ or } 16.$

SECOND SOLUTION

$$x^{\frac{1}{2}} - x^{\frac{1}{4}} = 6.$$

et $x^{\frac{1}{4}} = p$, then, $x^{\frac{1}{2}} = p^2$, and $p^2 - p = 6.$

$$\therefore p^2 - p - 6 = 0.$$

ctoring, $(p - 3)(p + 2) = 0.$

$$\therefore p = 3 \text{ or } -2;$$

s, $x^{\frac{1}{4}} = 3 \text{ or } -2.$

hence, $x = 81 \text{ or } 16.$

Solve the following equations :

3. $x^4 - 13x^2 + 36 = 0.$

4. $x^4 - 25x^2 + 144 = 0.$

5. $x^4 - 18x^2 + 32 = 0.$

6. $3x^4 + 5x^2 - 8 = 0.$

7. $5x^4 + 6x^2 - 11 = 0.$

8. $2x^4 - 8x^2 - 90 = 0.$

9. $x^{\frac{1}{2}} - 5x^{\frac{1}{4}} + 6 = 0.$

10. $x^{\frac{1}{2}} + 3x^{\frac{1}{4}} - 28 = 0.$

11. $x^{\frac{1}{3}} - 3x^{\frac{1}{6}} = -2.$

12. $x^{\frac{2}{3}} - x^{\frac{1}{3}} = 6.$

13. $x + 2\sqrt{x} = 3.$

14. $x^{\frac{2}{3}} - 2x^{\frac{1}{3}} = 3.$

15. $x^{\frac{4}{3}} = 10x^{\frac{2}{3}} - 9.$

16. $(x-3)^2 + 2(x-3)$

17. $(x^2+1)^2 + 4(x^2+1)$

18. $(x^2-4)^2 - 3(x^2-4)$

19. Solve the equation $x - 4x^{\frac{2}{3}} + 3x^{\frac{1}{3}} = 0.$

SOLUTION

Let $x^{\frac{1}{3}} = p$, then, $x^{\frac{2}{3}} = p^2$, and $x = p^3$.

Then, $p^3 - 4p^2 + 3p = 0.$

Factoring, $p(p^2 - 4p + 3) = 0.$

Whence, $p = 0$

or $p^2 - 4p + 3 = 0.$

Factoring, $(p-1)(p-3) = 0.$

Whence, $p = 1$ or $p = 3$.

That is, $x^{\frac{1}{3}} = 0, 1,$ or 3.

$\therefore x = 0, 1,$ or 27.

Solve:

20. $x^{\frac{3}{2}} - 4x - 5x^{\frac{1}{2}} = 0.$

21. $x - 3x^{\frac{3}{4}} + 2x^{\frac{1}{2}} = 0.$

22. $x + 2x^{\frac{5}{3}} - 3x^{\frac{4}{3}} = 0.$

23. $5x = x\sqrt{x} + 6\sqrt{x}.$

24. $3x = x\sqrt[3]{x} + 2\sqrt[3]{x^2}.$

25. $2x + \sqrt{x} = 15x\sqrt{x}.$

26. Given $x^2-7x+\sqrt{x^2-7x+18}=24$, to find the value of x.

SOLUTION

$$x^2-7x+\sqrt{x^2-7x+18}=24. \qquad (1)$$

Adding 18, $x^2-7x+18+\sqrt{x^2-7x+18}=42. \qquad (2)$

Put p for $(x^2-7x+18)^{\frac{1}{2}}$ and p^2 for $(x^2-7x+18)$. (3)

Then, $p^2+p-42=0. \qquad (4)$

Solving, $p=6$ or $-7. \qquad (5)$

That is, $\sqrt{x^2-7x+18}=6 \qquad (6)$

or $\sqrt{x^2-7x+18}=-7. \qquad (7)$

Squaring (6), $x^2-7x+18=36.$

Solving, $x=9$ or $-2.$

Squaring (7), $x^2-7x+18=49.$

Solving, $x=\frac{7}{2}\pm\frac{1}{2}\sqrt{173}.$

Hence, the roots of (1) are $x=9,\ -2,\ \frac{7}{2}\pm\frac{1}{2}\sqrt{173}.$

27. Solve the equation $x+2\sqrt{x+3}=21$.

28. Solve $x^2-3x+2\sqrt{x^2-3x+6}=18$.

29. Solve the equation $x^6-9x^3+8=0$.

SOLUTION

$$x^6-9x^3+8=0. \qquad (1)$$

Factoring, $(x^3-1)(x^3-8)=0. \qquad (2)$

Therefore, $x^3-1=0. \qquad (3)$

or $x^3-8=0. \qquad (4)$

If the values of x are found by transposing the known terms in (3) and (4) and then extracting the cube root of each member, only *one* value of x will be obtained from each equation. But if the equations are factored, *three* values of x are obtained for each.

Factoring (3), $(x-1)(x^2+x+1)=0, \qquad (5)$

and (4), $(x-2)(x^2+2x+4)=0. \qquad (6)$

Writing each factor equal to zero, and solving, we have:

From (5), $x=1,\ \frac{1}{2}(-1+\sqrt{-3}),\ \frac{1}{2}(-1-\sqrt{-3}). \qquad (7)$

From (6), $x=2,\ -1+\sqrt{-3},\ -1-\sqrt{-3}. \qquad (8)$

NOTE. — Since the values of x in (7) are obtained by factoring $x^3 - 1 = 0$, they may be regarded as the ***three cube roots of the number*** 1. Also, the values of x in (8) may be regarded as the ***three cube roots of the number*** 8 (§ 286).

Solve:

30. $x^6 - 28x^3 + 27 = 0$.

31. $x^4 - 16 = 0$.

32. Find the three cube roots of -1.

33. Find the three cube roots of -8.

34. Solve the equation $x^4 + 4x^3 - 8x + 3 = 0$.

FIRST SOLUTION

Extracting the square root of the first member as far as possible,

$$
\begin{array}{r|l}
 & x^4 + 4x^3 - 8x + 3 \,\underline{|\, x^2 + 2x - 2} \\
 & x^4 \\
\hline
2x^2 + 2x & 4x^3 \\
 & \underline{4x^3 + 4x^2} \\
\hline
2x^2 + 4x - 2 & -4x^2 - 8x + 3 \\
 & \underline{-4x^2 - 8x + 4} \\
 & -1
\end{array}
$$

Since the first member lacks 1 of being a perfect square, the square may be completed by adding 1 to each member, which gives the following equation:

$$x^4 + 4x^3 - 8x + 4 = 1.$$

Extracting the square root, $x^2 + 2x - 2 = \pm 1$.

$$\therefore x^2 + 2x - 3 = 0, \text{ and } x^2 + 2x - 1 = 0.$$

Solving, $x = 1,\ -3,\ -1 \pm \sqrt{2}$.

SECOND SOLUTION

If $4x^2$ is added to the first two terms of the given equation, the resulting trinomial, $x^4 + 4x^3 + 4x^2$, will be a perfect square. Adding $4x^2$ and subtracting $4x^2$ does not change the first member.

Then, $x^4 + 4x^3 + 4x^2 - 4x^2 - 8x + 3 = 0$;

whence, $(x^2 + 2x)^2 - 4(x^2 + 2x) + 3 = 0$.

Factoring, $(x^2 + 2x - 3)(x^2 + 2x - 1) = 0$.

Solving, $x = 1,\ -3,\ -1 \pm \sqrt{2}$.

Third Solution

By applying the factor theorem (§ 164), the factors of the first member are found to be $x-1$, $x+3$, and x^2+2x-1; that is,

$$(x-1)(x+3)(x^2+2x-1)=0.$$

Solving, $$x=1,\ -3,\ -1\pm\sqrt{2}.$$

Solve:

35. $x^4+2x^3-x=30.$

36. $x^4-4x^3+8x=-3.$

37. $x^4-2x^3+x=132.$

38. $x^4-6x^3+27x=10.$

39. $x^4+2x^3+5x^2+4x-60=0.$

40. $x^4+6x^3+7x^2-6x-8=0.$

41. $x^4-6x^3+15x^2-18x+8=0.$

42. $\dfrac{x^2}{x+1}+\dfrac{x+1}{x^2}=\dfrac{25}{12}.$

Suggestion. — Since the second term is the *reciprocal* of the first, put p for the first term and $\dfrac{1}{p}$ for the second.

Then, $$p+\frac{1}{p}=\frac{25}{12}.$$

43. $\dfrac{x^2+x}{2}+\dfrac{2}{x^2+x}=2.$

44. $\dfrac{x^2+1}{4}+\dfrac{4}{x^2+1}=\dfrac{5}{2}.$

45. $\dfrac{x+2}{x^2+4}+\dfrac{2(x^2+4)}{x+2}=\dfrac{51}{5}.$

46. $\dfrac{x^2+1}{x-1}-\dfrac{4(x-1)}{x^2+1}=\dfrac{21}{5}.$

Solve the following miscellaneous equations:

47. $x^3+8x^{\frac{3}{2}}-9=0.$

48. $x^{\frac{3}{2}}+x^{\frac{3}{4}}-2=0.$

49. $\sqrt[4]{x}+3\sqrt{x}=30.$

50. $ax^{2n}+bx^n+c=0.$

51. $x^{\frac{7}{2}}-x^2-12x^{\frac{1}{2}}=0.$

52. $x^4=8x+7x^2\sqrt{x}.$

53. $x-5+2\sqrt{x-5}=8.$

54. $x+10=2\sqrt{x+10}+5.$

55. $x-3=21-4\sqrt{x-3}.$

56. $2x-3\sqrt{2x+5}=-5.$

57. $2x - 6\sqrt{2x - 1} = 8.$

58. $x = 11 - 3\sqrt{x + 7}.$

59. $x^3 + x\sqrt{x} - 72 = 0.$

60. $x^{-\frac{2}{3}} - 5x^{-\frac{1}{3}} + 4 = 0.$

61. $x^2 - 5x + 2\sqrt{x^2 - 5x - 2} = 10.$

62. $x^2 - x - \sqrt{x^2 - x + 4} - 8 = 0.$

63. $x^2 - 5x + 5\sqrt{x^2 - 5x + 1} = 49.$

64. $(x^2 - 2x)^2 - 2(x^2 - 2x) = 3.$

65. $(x^2 - x)^2 - (x^2 - x) - 132 = 0.$

66. $\left(\frac{12}{x} - 1\right)^2 + 8\left(\frac{12}{x} - 1\right) = 33.$

67. $\left(x + \frac{1}{x}\right)^2 - 2\left(x + \frac{1}{x}\right) = \frac{5}{4}.$

68. $\left(\frac{1 + x^2}{x}\right)^2 + 2\left(\frac{1 + x^2}{x}\right) = 8.$

69. $\left(x - \frac{1}{x}\right)^2 + \frac{5}{6}\left(x - \frac{1}{x}\right) = 1.$

70. $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0.$

71. $16x^4 - 8x^3 - 31x^2 + 8x + 15 = 0.$

72. $4x^4 - 4x^3 - 7x^2 + 4x + 3 = 0.$

73. $x^2 + x + 1 - \frac{1}{x^2 + x + 1} = \frac{8}{3}.$

74. $x^2 - 2x + \frac{4}{x^2 - 2x + 1} = 4.$

75. $x^2 - 3x + \frac{2}{x^2 - 3x + 2} = 1.$

76. $x^2 - x + \frac{2}{x^2 - x - 4} = 7.$

77. $\frac{x}{x^2 - 1} + \frac{x^2 - 1}{x} = -\frac{13}{6}.$

78. $\frac{1}{1 + x + x^2} + \frac{2}{\sqrt{1 + x + x^2}} - 3 = 0.$

SIMULTANEOUS EQUATIONS INVOLVING QUADRATICS

402. Two simultaneous *quadratic* equations involving two unknown numbers generally lead to equations of the fourth degree, and therefore they cannot be solved usually by quadratic methods.

However, there are some simultaneous equations involving quadratics that may be solved by quadratic methods, as shown in the following cases.

403. When one equation is simple and the other of higher degree.

Equations of this class may be solved by finding the value of one unknown number in terms of the other in the simple equation, and then substituting that value in the other equation.

EXERCISES

404. **1.** Solve the equations $\begin{cases} x+y=7, \\ 3x^2+y^2=43. \end{cases}$

SOLUTION

$$x+y=7. \tag{1}$$

$$3x^2+y^2=43. \tag{2}$$

From (1), $\quad y=7-x. \quad (3)$

Substituting in (2), $\quad 3x^2+(7-x)^2=43. \quad (4)$

Solving, $\quad x=3 \text{ or } \frac{1}{2}. \quad (5)$

Substituting 3 for x in (3), $\quad y=4. \quad (6)$

Substituting $\frac{1}{2}$ for x in (3), $\quad y=\frac{13}{2}. \quad (7)$

That is, x and y each have two values $\begin{cases} \text{when } x=3,\ y=4, \\ \text{when } x=\frac{1}{2},\ y=\frac{13}{2}. \end{cases}$

Solve the following equations:

2. $\begin{cases} x^2+y^2=20, \\ x=2y. \end{cases}$

3. $\begin{cases} 10x+y=3xy, \\ y-x=2. \end{cases}$

4. $\begin{cases} x^2+xy=12, \\ x-y=2. \end{cases}$

5. $\begin{cases} m^2-3n^2=13, \\ m-2n=1. \end{cases}$

6. $\begin{cases} x = 6 - y, \\ x^3 + y^3 = 72. \end{cases}$

7. $\begin{cases} xy(x - 2y) = 10, \\ xy = 10. \end{cases}$

8. $\begin{cases} 3x(y + 1) = 12, \\ 3x = 2y. \end{cases}$

9. $\begin{cases} 3rs - 10r = s, \\ 2 - s = -r. \end{cases}$

405. An equation that is not affected by interchanging the unknown numbers involved is called a **symmetrical equation**.

$2x^2 + xy + 2y^2 = 4$ and $x^2 + y^2 = 9$ are symmetrical equations.

406. **When both equations are symmetrical.**

Though equations of this class may be solved usually by substitution, as in §§ 403, 404, it is preferable to find values for $x + y$ and $x - y$ and then solve these simple equations for x and y.

EXERCISES

407. 1. Solve the equations $\begin{cases} x + y = 7, \\ xy = 10. \end{cases}$

SOLUTION

$$x + y = 7. \quad (1)$$

$$xy = 10. \quad (2)$$

Squaring (1), $x^2 + 2xy + y^2 = 49$. (3)

Multiplying (2) by 4, $4xy = 40$. (4)

Subtracting (4) from (3), $x^2 - 2xy + y^2 = 9$. (5)

Extracting the square root, $x - y = \pm 3$. (6)

From (1) + (6), $x = 5$ or 2.

From (1) − (6), $y = 2$ or 5.

2. Solve the equations $\begin{cases} x^2 + y^2 = 25, \\ x + y = 7. \end{cases}$

SUGGESTION. — From the square of the second equation subtract the *first equation*; then subtract this result from the first equation and proceed as in exercise 1.

3. **Solve the equations** $\begin{cases} x^4 + y^4 = 97, \\ x + y = 1. \end{cases}$

SOLUTION

$$x^4 + y^4 = 97. \quad (1)$$

$$x + y = 1. \quad (2)$$

Raising (2) to the fourth power,

$$x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 = 1. \quad (3)$$

Subtracting (1) from (3), $4x^3y + 6x^2y^2 + 4xy^3 = -96.$ (4)

Dividing by 2, $2x^3y + 3x^2y^2 + 2xy^3 = -48.$ (5)

$2xy \times$ square of (2), $2x^3y + 4x^2y^2 + 2xy^3 = 2xy.$ (6)

Subtracting (5) from (6), $x^2y^2 - 2xy = 48.$ (7)

Solving for xy, $xy = -6$ or $8.$ (8)

Equations (2) and (8) give two pairs of simultaneous equations,

$$\begin{cases} x + y = 1 \\ xy = -6 \end{cases} \text{ and } \begin{cases} x + y = 1 \\ xy = 8 \end{cases}$$

Solving as in exercise 1, the corresponding values of x and y are

$$\begin{cases} x = 3; \ -2; \ \tfrac{1}{2}(1 + \sqrt{-31}); \ \tfrac{1}{2}(1 - \sqrt{-31}); \\ y = -2; \ 3; \ \tfrac{1}{2}(1 - \sqrt{-31}); \ \tfrac{1}{2}(1 + \sqrt{-31}). \end{cases}$$

Solve the following equations:

4. $\begin{cases} x^2 + y^2 = 50, \\ xy = 7. \end{cases}$

5. $\begin{cases} x + y = 8, \\ x^2 + y^2 = 34. \end{cases}$

6. $\begin{cases} x + y = 9, \\ x^3 + y^3 = 243. \end{cases}$

7. $\begin{cases} x + y = 8, \\ x^2 + xy + y^2 = 49. \end{cases}$

8. $\begin{cases} x^2 + xy + y^2 = 31, \\ x^2 + y^2 = 26. \end{cases}$

9. $\begin{cases} x^2 + y^2 = 8. \\ x^2 - xy + y^2 = 4. \end{cases}$

10. $\begin{cases} x^2 + y^2 = 13, \\ x + y + xy = 11. \end{cases}$

11. $\begin{cases} x^2 + 3xy + y^2 = 31, \\ xy = 6. \end{cases}$

12. $\begin{cases} x^2 + y^2 = 100, \\ (x + y)^2 = 196. \end{cases}$

13. $\begin{cases} x + xy + y = 19, \\ x^2y^2 = 144. \end{cases}$

14. $\begin{cases} x^4 + y^4 = 17, \\ x + y = 3. \end{cases}$ 15. $\begin{cases} x^4 + x^2y^2 + y^4 \\ x^2 + xy + y^2 = \end{cases}$

408. An equation *all* of whose terms are of the sam with respect to the unknown numbers is called a **hom equation.**

$x^2 - xy = y^2$ and $3x^3 + y^3 = 0$ are homogeneous equations.

An equation like $x^2 - xy + y^2 = 21$ is said to be **hom in the unknown terms.**

409. When both equations are quadratic, one being homo

In this case elimination may always be effected by s tion, for by dividing the homogeneous equation through becomes a quadratic in $\frac{x}{y}$. The two values of $\frac{x}{y}$ obtain this equation give two *simple* equations in x and y, which may be combined with the remaining quadrati tion as in §§ 403, 404.

Thus, $ax^2 + bxy + cy^2 = 0$ is the general form of th geneous equation in which a, b, and c are known numb

Dividing by y^2, we have $a\left(\frac{x}{y}\right)^2 + b\left(\frac{x}{y}\right) + c = 0$, a quadra

EXERCISES

410. 1. Solve the equations $\begin{cases} x^2 + 3x - y = 5, \\ 5x^2 + 4xy - y^2 = 0. \end{cases}$

SOLUTION

$$x^2 + 3x - y = 5.$$

$$5x^2 + 4xy - y^2 = 0.$$

Dividing (2) by y^2, $5\left(\frac{x}{y}\right)^2 + 4\left(\frac{x}{y}\right) - 1 = 0$, a quadratic in $\frac{x}{y}$ wl be solved by factoring or by completing the square.

To avoid fractions, however, (2) may be factored at once; th

$$(x + y)(5x - y) = 0.$$

$$\therefore y = -x \text{ or } 5x.$$

Substituting $-x$ for y in (1), simplifying, etc.,

$$x^2 + 4x = 5.$$

Solving, $$x = 1 \text{ or } -5. \tag{3}$$

$$\therefore y = -x = -1 \text{ or } 5. \tag{4}$$

Substituting $5x$ for y in (1), simplifying, etc.,

$$x^2 - 2x = 5.$$

Solving, $$x = 1 + \sqrt{6} \text{ or } 1 - \sqrt{6}. \tag{5}$$

$$\therefore y = 5x = 5(1 + \sqrt{6}) \text{ or } 5(1 - \sqrt{6}). \tag{6}$$

Hence, from (3), (4), (5), and (6) the roots of the given equation are

$$\begin{cases} x = 1; & -5; & 1 + \sqrt{6}; & 1 - \sqrt{6}; \\ y = -1; & 5; & 5(1 + \sqrt{6}); & 5(1 - \sqrt{6}). \end{cases}$$

Solve the following equations:

2. $\begin{cases} 2x^2 - 3y - y^2 = 8, \\ 6x^2 - 5xy - 6y^2 = 0. \end{cases}$

3. $\begin{cases} 5x^2 + 8xy - 4y^2 = 0, \\ xy + 2y^2 = 60. \end{cases}$

4. $\begin{cases} 2x^2 - xy - y^2 = 0, \\ 4x^2 + 4xy + y^2 = 36. \end{cases}$

5. $\begin{cases} 6x^2 + xy - 12y^2 = 0, \\ x^2 + xy - y = 1. \end{cases}$

6. $\begin{cases} 3x^2 - 7xy - 40y^2 = 0, \\ x^2 - xy - 12y^2 = 8. \end{cases}$

7. $\begin{cases} x^2 - xy - y^2 = 20, \\ 3x^2 - 13xy + 12y^2 = 0. \end{cases}$

8. $\begin{cases} 3x^2 - 7xy + 4y^2 = 0, \\ 5x^2 - 7xy + 3y^2 = 4. \end{cases}$

9. $\begin{cases} x^2 + y^2 + x - y = 12, \\ 3x^2 + 2xy - y^2 = 0. \end{cases}$

411. When both equations are quadratic and homogeneous in the unknown terms.

Either of the following methods may be employed in this case:

Substitute vy for x, solve for y^2 in each equation, and compare the values of y^2 thus found, forming a quadratic in v.

Or, eliminate the absolute term, forming a homogeneous equation; *then proceed* as in §§ 409, 410.

EXERCISES

412. 1. Solve the equations $\begin{cases} x^2 - xy + y^2 = 21, \\ y^2 - 2xy = -15. \end{cases}$

First Solution

$$x^2 - xy + y^2 = 21. \tag{1}$$

$$y^2 - 2xy = -15. \tag{2}$$

Assume $$x = vy. \tag{3}$$

Substituting in (1), $$v^2y^2 - vy^2 + y^2 = 21. \tag{4}$$

Substituting in (2), $$y^2 - 2vy^2 = -15. \tag{5}$$

Solving (4) for y^2, $$y^2 = \frac{21}{v^2 - v + 1}. \tag{6}$$

Solving (5) for y^2, $$y^2 = \frac{15}{2v - 1}. \tag{7}$$

Comparing the values of y^2, $$\frac{15}{2v-1} = \frac{21}{v^2 - v + 1}. \tag{8}$$

Clearing, etc., $$5v^2 - 19v + 12 = 0. \tag{9}$$

Factoring, $$(v - 3)(5v - 4) = 0. \tag{10}$$

$$\therefore v = 3 \text{ or } \tfrac{4}{5}. \tag{11}$$

Substituting 3 for v in (7) or in (6), $y = \pm\sqrt{3}$
and since $x = vy$, $x = \pm 3\sqrt{3}.$ (12)

Substituting $\frac{4}{5}$ for v in (7) or in (6), $y = \pm 5$
and since $x = vy$, $x = \pm 4.$ (13)

When the double sign is used, as in (12) and in (13), it is understood that the roots shall be associated by taking the *upper* signs together and the *lower* signs together.

Hence, $\begin{cases} x = 3\sqrt{3}; & -3\sqrt{3}; & 4; & -4; \\ y = \sqrt{3}; & -\sqrt{3}; & 5; & -5. \end{cases}$

Suggestion for Second Solution. — Multiplying the first equation by 5 and the second by 7, and adding the results, we eliminate the absolute term and obtain the homogeneous equation

$$5x^2 - 19xy + 12y^2 = 0,$$

which may be solved with either of the given equations, as in exercise *1*, § 410.

Solve the following equations:

2. $\begin{cases} xy + 3y^2 = 20, \\ x^2 - 3xy = -8. \end{cases}$

3. $\begin{cases} x^2 + xy = 12, \\ xy + 2y^2 = 5. \end{cases}$

4. $\begin{cases} x^2 + 2y^2 = 44, \\ xy - y^2 = 8. \end{cases}$

5. $\begin{cases} x(x-y) = 6, \\ x^2 - y^2 = 3. \end{cases}$

6. $\begin{cases} x^2 - xy - y^2 = 20, \\ x^2 - 3xy + 2y^2 = 8. \end{cases}$

7. $\begin{cases} x^2 - xy + y^2 = 21, \\ x^2 + 2y^2 = 27. \end{cases}$

8. $\begin{cases} 2x^2 - 3xy + 2y^2 = 100, \\ x^2 - y^2 = 75. \end{cases}$

9. $\begin{cases} x^2 - 5xy + 3y^2 = 8, \\ 3x^2 + xy + y^2 = 24. \end{cases}$

413. Special devices.

Many systems of simultaneous equations that belong to one or more of the preceding classes, and many that belong to none of them, may be solved readily by *special devices*. It is impossible to lay down any fixed line of procedure, but the object often aimed at is to find values for *any two* of the expressions, $x + y$, $x - y$, and xy, from which the values of x and y may be obtained. Various manipulations are resorted to in attaining this object, according to the forms of the given equations.

EXERCISES

414. 1. Solve the equations $\begin{cases} x^2 + xy = 12, \\ xy + y^2 = 4. \end{cases}$

SOLUTION

$$x^2 + xy = 12. \tag{1}$$

$$xy + y^2 = 4. \tag{2}$$

Adding (1) and (2), $$x^2 + 2xy + y^2 = 16. \tag{3}$$

$$\therefore\ x + y = +4 \text{ or } -4. \tag{4}$$

Subtracting (2) from (1), $$x^2 - y^2 = 8. \tag{5}$$

Dividing (5) by (4), $$x - y = +2 \text{ or } -2. \tag{6}$$

Combining (4) and (6), $$x = 3 \text{ or } -3;\ y = 1 \text{ or } -1.$$

NOTE. — The first value of $x - y$ corresponds only to the first value of $x + y$, and the second value only to the second value.

Consequently, there are only two pairs of values of x and y.

Observe that the given equations belong to the class treated in § 411. The special device adopted here, however, gives a much neater and simpler solution than either of the methods presented in that case.

2. Solve the equations $\begin{cases} x^2+y^2+x+y=14, \\ xy=3. \end{cases}$

SOLUTION

$$x^2+y^2+x+y=14. \tag{1}$$

$$xy=3. \tag{2}$$

Adding twice the second equation to the first,

$$x^2+2xy+y^2+x+y=20.$$

Completing the square, $(x+y)^2+(x+y)+(\frac{1}{2})^2=20\frac{1}{4}$.

Extracting the square root, $x+y+\frac{1}{2}=\pm\frac{9}{2}$.

$$\therefore\ x+y=4 \text{ or } -5. \tag{3}$$

Equations (2) and (3) give two pairs of simultaneous equations,

$$\begin{cases} x+y=4 \\ xy=3 \end{cases} \text{ and } \begin{cases} x+y=-5 \\ xy=3 \end{cases}$$

Solving, the corresponding values of x and y are found to be

$$\begin{cases} x=3;\ 1;\ \frac{1}{2}(-5+\sqrt{13});\ \frac{1}{2}(-5-\sqrt{13}); \\ y=1;\ 3;\ \frac{1}{2}(-5-\sqrt{13});\ \frac{1}{2}(-5+\sqrt{13}). \end{cases}$$

Symmetrical except as to sign.—When one of the equations is symmetrical and the other would be symmetrical if one or more of its signs were changed, or when both equations are of the latter type, the system may be solved by the methods used for symmetrical equations (§ 406).

3. Solve the equations $\begin{cases} x^2+y^2=53, \\ x-y=5. \end{cases}$

SOLUTION

$$x^2+y^2=53. \tag{1}$$

4. Solve the equations $\begin{cases} \dfrac{1}{x^2} + \dfrac{1}{y^2} = 74, \\ \dfrac{1}{x} - \dfrac{1}{y} = 2. \end{cases}$

SUGGESTION. — Square the second equation and proceed exactly as in xercise 3. Solving at first for $\frac{1}{x}$ and $\frac{1}{y}$ instead of for x and y, the result is

$$\frac{1}{x} = 7 \text{ or } -5, \qquad (1)$$

nd

$$\frac{1}{y} = 5 \text{ or } -7. \qquad (2)$$

Solving (1), $x = \frac{1}{7}$ or $-\frac{1}{5}$.

Solving (2), $y = \frac{1}{5}$ or $-\frac{1}{7}$.

NOTE. — It is sometimes convenient to begin by solving for expressions ther than x and y, as $\sqrt{xy}$, $\sqrt{x+y}$, etc.

Whether the equations are symmetrical or symmetrical ex- ept for the sign, it is often advantageous to substitute $u+v$ or x and $u-v$ for y.

5. Solve the equations $\begin{cases} x^4 + y^4 = 82, \\ x - y = 2. \end{cases}$

SOLUTION

$$x^4 + y^4 = 82. \qquad (1)$$

$$x - y = 2. \qquad (2)$$

Assume $x = u + v$, (3)

and $y = u - v$. (4)

Substituting these values in (1),

$$u^4 + 4u^3v + 6u^2v^2 + 4uv^3 + v^4$$
$$+ u^4 - 4u^3v + 6u^2v^2 - 4uv^3 + v^4 = 82, \qquad (5)$$

and in (2), $2v = 2$. (6)

Dividing (5) by 2, $u^4 + 6u^2v^2 + v^4 = 41$. (7)

Dividing (6) by 2, $v = 1$. (8)

Substituting 1 for v in (7) and solving, $u = \pm 2$ or $\pm\sqrt{-10}$. (9)

Hence, substituting (8) and (9) in (3) and (4), the corresponding values of x and y are found to be

$$\begin{cases} x = 3; & -1; & 1 + \sqrt{-10}; & 1 - \sqrt{-10}; \\ y = 1; & -3; & -1 + \sqrt{-10}; & -1 - \sqrt{-10}. \end{cases}$$

NOTE. — The given system of equations may be solved also by the method of *exercise 3, § 407.*

Division of one equation by the other. — The reduction of equations of higher degree to quadratics is often effected by dividing one of the given equations by the other, *member by member.*

6. Solve the equations $\begin{cases} x^4 + x^2y^2 + y^4 = 91, \\ x^2 - xy + y^2 = 7. \end{cases}$

SOLUTION

$$x^4 + x^2y^2 + y^4 = 91. \quad (1)$$

$$x^2 - xy + y^2 = 7. \quad (2)$$

Dividing (1) by (2), $x^2 + xy + y^2 = 13.$ (3)

Subtracting (2) from (3), $2xy = 6;$

whence, $xy = 3.$ (4)

Adding (4) and (3), $x^2 + 2xy + y^2 = 16.$ (5)

Subtracting (4) from (2), $x^2 - 2xy + y^2 = 4.$ (6)

Extracting the square root of (5), $x + y = 4 \text{ or } -4.$ (7)

Extracting the square root of (6), $x - y = 2 \text{ or } -2.$ (8)

Solving these simultaneous equations in (7) and (8),

$$\begin{cases} x = 3; \ 1; \ -1; \ -3; \\ y = 1; \ 3; \ -3; \ -1. \end{cases}$$

NOTE. — Since (7) and (8) have been derived independently, with the first value of $x + y$ we associate *each* value of $x - y$ in succession, and with the second value of $x + y$ each value of $x - y$ in succession, *in the same order.*

Consequently, there are four pairs of values of x and y.

7. Solve the equations $\begin{cases} x^3 - y^3 = 26, \\ x - y = 2. \end{cases}$

SUGGESTION. — Dividing the first equation by the second,

$$x^2 + xy + y^2 = 13.$$

Therefore, solve the system,

$$\begin{cases} x^2 + xy + y^2 = 13, \\ x - y = 2, \end{cases}$$

instead of the given system.

NOTE. — It is sometimes the case that a root is removed when one equation is divided by the other, member by member.

Observe that the given system may be solved by the method used in *exercise 5, but* the solution suggested here is briefer and simpler.

Elimination of similar terms. — When the equations are quadratic and each is homogeneous except for one term, if these excepted terms are similar in the two equations, they may be eliminated and the solution of the system be made to depend on the case of § 409.

Some equations belonging to this class, namely, those that are homogeneous except for the *absolute term*, have been treated in § 411.

8. Solve the equations $\begin{cases} x^2 + 2xy = \frac{5}{2}y, \\ 2x^2 - xy + y^2 = 2y. \end{cases}$

SUGGESTION. — Multiplying the first equation by 4 and the second by 5,

$$4x^2 + 8xy = 10y, \qquad (a)$$

and

$$10x^2 - 5xy + 5y^2 = 10y. \qquad (b)$$

Subtracting (*a*) from (*b*),

$$6x^2 - 13xy + 5y^2 = 0, \text{ a homogeneous equation.}$$

Therefore, solve the system,

$$\begin{cases} 6x^2 - 13xy + 5y^2 = 0, \\ 2x^2 - xy + y^2 = 2y, \end{cases}$$

instead of the given system, using the method of § 409.

Instead of eliminating terms below the second degree, as in exercise 8, in certain systems it is advantageous to eliminate similar terms of the second degree.

9. Solve the equations $\begin{cases} xy + x = 35, \\ xy + y = 32. \end{cases}$

SOLUTION

$$xy + x = 35. \qquad (1)$$

$$xy + y = 32. \qquad (2)$$

Subtracting (2) from (1), $x - y = 3$;

hence, $y = x - 3.$ (3)

Substituting (3) in (1), $x(x - 3) + x = 35,$

$$x^2 - 2x = 35.$$

Solving, $x = 7 \text{ or } -5.$ (4)

Substituting (4) in (3), $y = 4 \text{ or } -8.$

Solve, using the methods illustrated in exercises 1–9:

10. $\begin{cases} m^2 + mn = 2, \\ mn + n^2 = -1. \end{cases}$

11. $\begin{cases} p^2 + q^2 + p + q = 36, \\ pq = -15. \end{cases}$

12. $\begin{cases} a^2 + b^2 = 130, \\ a - b = 2. \end{cases}$

13. $\begin{cases} \dfrac{1}{x^2} + \dfrac{1}{y^2} = 13, \\ \dfrac{1}{x} - \dfrac{1}{y} = 1. \end{cases}$

14. $\begin{cases} x^4 + y^4 = 17, \\ x - y = 1. \end{cases}$

15. $\begin{cases} c^4 + c^2d^2 + d^4 = 3, \\ c^2 - cd + d^2 = 3. \end{cases}$

16. $\begin{cases} r^3 - s^3 = 54, \\ r - s = 6. \end{cases}$

17. $\begin{cases} x^2 + 2xy = 7y, \\ 2x^2 - xy + y^2 = 8y \end{cases}$

18. $\begin{cases} xy + x = 32, \\ xy + y = 27. \end{cases}$

19. $\begin{cases} 2x^2 - 3y^2 = 5, \\ 3x^2 - 2y^2 = 30. \end{cases}$

415. All the solutions in §§ 403–414 are but illustrations methods that are important because they are often applicab The student is urged to use his ingenuity in devising otl methods or modifications of these whenever the given syst does not yield readily to the devices illustrated, or whene a simpler solution would result.

EXERCISES

416. Solve the following miscellaneous systems of equatio

1. $\begin{cases} x + y = 3, \\ xy = 2. \end{cases}$

2. $\begin{cases} 5x^2 - 4y^2 = 44, \\ 4x^2 - 5y^2 = 19. \end{cases}$

3. $\begin{cases} 1 + x = y, \\ x^2 + y^2 = 61. \end{cases}$

4. $\begin{cases} x^2 - xy = 48, \\ xy - y^2 = 12. \end{cases}$

5. $\begin{cases} a + ab + 28 = 0, \\ b + ab + 40 = 0. \end{cases}$

6. $\begin{cases} x^2 - 3xy = 8x, \\ 2x^2 - xy + y^2 = 8x. \end{cases}$

7. $\begin{cases} x^4 + x^2y^2 + y^4 = 21, \\ x^2 - xy + y^2 = 7. \end{cases}$

8. $\begin{cases} x^2 + y^2 + x + y = 26, \\ xy = -12. \end{cases}$

$\begin{cases} x^2 + y^2 = 40, \\ xy = 12. \end{cases}$

$\begin{cases} x^2 + xy = -6, \\ xy + y^2 = 15. \end{cases}$

$\begin{cases} x^3 + y^3 = 28, \\ x + y = 4. \end{cases}$

$\begin{cases} x^4 + y^4 = 82, \\ x + y = 4. \end{cases}$

$\begin{cases} x^4 + y^4 = 17, \\ x - y = -3. \end{cases}$

4. $\begin{cases} xy + x^2 = 44, \\ xy + y^2 = -28. \end{cases}$

5. $\begin{cases} x^2 + 4x + 3y = -1, \\ 2x^2 + 5xy + 2y^2 = 0. \end{cases}$

16. $\begin{cases} \dfrac{1}{m} + \dfrac{1}{n} = \dfrac{1}{2}, \\ \dfrac{1}{mn} - \dfrac{1}{18} = 0. \end{cases}$

17. $\begin{cases} x^2 - xy = 6, \\ x^2 + y^2 = 61. \end{cases}$

18. $\begin{cases} x^2 + xy = 77, \\ xy - y^2 = 12. \end{cases}$

19. $\begin{cases} 2x - y = 2, \\ 2x^2 + y^2 = \frac{3}{2}. \end{cases}$

20. $\begin{cases} x^2 + xy + y^2 = 19, \\ x^3 - y^3 = 19. \end{cases}$

21. $\begin{cases} x^2 + 3xy = y^2 + 23, \\ x + 3y = 9. \end{cases}$

22. $\begin{cases} 4xy = 96 - x^2y^2, \\ x + y = 6. \end{cases}$

23. $\begin{cases} x^2 - xy = 8, \\ xy + y^2 = 12. \end{cases}$

24. $\begin{cases} x(x + y) = x, \\ y(x - y) = -1. \end{cases}$

25. $\begin{cases} x^2 + 3xy - y^2 = 43, \\ x + 2y = 10. \end{cases}$

26. $\begin{cases} 2x^2 + 3xy + y^2 = 20, \\ 5x^2 + 4y^2 = 41. \end{cases}$

27. $\begin{cases} 2xy - y^2 = 12, \\ 3xy + 5x^2 = 104. \end{cases}$

28. $\begin{cases} x^2 + xy + y^2 = 151, \\ x^2 + y^2 = 106. \end{cases}$

29. $\begin{cases} 1 + x = y, \\ 1 + x^3 = \dfrac{y^3}{4}. \end{cases}$

30. $\begin{cases} x^4 - y^4 = 369, \\ x^2 - y^2 = 9. \end{cases}$

31. $\begin{cases} x^2 + xy + y^2 = 84, \\ x - \sqrt{xy} + y = 6. \end{cases}$

32. $\begin{cases} 4x^2 - 2xy + y^2 = 13, \\ 8x^3 + y^3 = 65. \end{cases}$

33. $\begin{cases} 6x^2 + 6y^2 = 13xy, \\ x^2 - y^2 = 20. \end{cases}$

34. $\begin{cases} x^2 + y^2 - 3(x + y) = 8 \\ x + y + xy = 11. \end{cases}$

35. $\begin{cases} x^3 - y^3 = 37, \\ xy(y - x) = -12. \end{cases}$

36. $\begin{cases} x + y = 25, \\ \sqrt{x} + \sqrt{y} = 7. \end{cases}$

37. $\begin{cases} x^3 + y^3 = 225\,y, \\ x^2 - y^2 = 75. \end{cases}$

38. $\begin{cases} x^2 + y^2 = 3\,xy + 5, \\ x^4 + y^4 = 2. \end{cases}$

39. $\begin{cases} 3\,xy + 2\,x + y = 25, \\ 9\,x^2 - 4\,y^2 = 0. \end{cases}$

40. $\begin{cases} x^2 - 7\,xy + 12\,y^2 = 0, \\ xy + 3\,y = 2\,x + 21. \end{cases}$

41. $\begin{cases} (x + y)(x^2 + y^2) = 65, \\ (x - y)(x^2 - y^2) = 5. \end{cases}$

42. $\begin{cases} x^2 + y = x - y^2 + 42, \\ xy = 20. \end{cases}$

43. $\begin{cases} x + y + 2\sqrt{x + y} = 24, \\ x - y + 3\sqrt{x - y} = 10. \end{cases}$

44. $\begin{cases} x^2 + y^2 + 6\sqrt{x^2 + y^2} = 55, \\ x^2 - y^2 = 7. \end{cases}$

45. $\begin{cases} x^2 - 6\,xy + 9\,y^2 + 2\,x - 6\,y - 8 = 0, \\ x^2 + 4\,xy + 4\,y^2 - 4\,x - 8\,y - 21 = 0. \end{cases}$

SUGGESTION. — The equations may be written in the quadratic form

Thus, $\begin{cases} (x - 3\,y)^2 + 2(x - 3\,y) - 8 = 0, \\ (x + 2\,y)^2 - 4(x + 2\,y) - 21 = 0. \end{cases}$

46. Solve $\left.\begin{cases} x^2 - xy = a^2 + b^2 \\ xy - y^2 = 2\,ab \end{cases}\right\}$ for x and y.

47. Solve $\left.\begin{cases} x - 2\,y = 2\,(a + b) \\ xy + 2\,y^2 = 2\,b\,(b - a) \end{cases}\right\}$ for x and y.

48. Solve $\left.\begin{cases} x^3 + y^3 = 2\,a\,(a^2 + 3\,b^2) \\ x^2y + xy^2 = 2\,a\,(a^2 - b^2) \end{cases}\right\}$ for x and y.

49. Solve $\left.\begin{cases} s = \frac{1}{2}\,at^2 \\ v = at \end{cases}\right\}$ for a and t.

50. Solve $\left.\begin{cases} s = 6\,t + \frac{1}{2}\,at^2 \\ v = at \end{cases}\right\}$ for v and t.

Problems

. 1. The sum of two numbers is 12, and their product is Vhat are the numbers?

The sum of two numbers is 17, and the sum of their s is 157. What are the numbers?

The difference of two numbers is 1, and the difference of ubes is 91. What are the numbers?

The sum of two numbers is 82, and the sum of their roots is 10. What are the numbers?

It takes 52 rods of fence to inclose a rectangular garden ning 1 acre. How long and how wide is the garden?

The product of two numbers is 59 greater than their sum. e sum of their squares is 170. What are the numbers?

If 63 is subtracted from a certain number expressed by igits, its digits will be transposed; and if the number is lied by the sum of its digits, the product will be 729. is the number?

A man expended $ 6.00 for canvas. Had it cost 4 cents er yard, he would have received 5 yards more. How yards did he buy, and at what price per yard?

Mr. Fuller paid $ 2.25 for some Italian olive oil, and $ 2.00 gallon less of French olive oil, which cost $ .50 more per . How much of each kind did he buy and at what price?

In papering a room, 18 yards of border were required, 40 yards of paper $\frac{1}{2}$ yard wide were needed to cover iling exactly. Find the length and breadth of the room.

An Illinois farmer raised broom corn and pressed the ounds of brush into bales. If he had made each bale nds heavier, he would have had 1 bale less. How many lid he press and what was the weight of each?

The total area of a window screen whose length is 4 inches r than its width is 10 square feet. The area inside the *frame is 8* square feet. Find the width of the frame.

13. A boy has a large blotter, 4 inches longer than it is wide, and 480 square inches in area. He wishes to cut away enough to leave a square 256 square inches in area. How many inches must he cut from the length and from the width?

14. A man bought 4 more loads of sand than of gravel, paying \$.50 less per load for sand than for gravel. The sand cost him \$9.00 and the gravel \$10.00. What quantities of each did he buy? What prices did he pay?

15. The course for a 36-mile yacht race is the perimeter of a right triangle, one leg of which is 3 miles longer than the other. How long is each side of the course?

16. A rectangular skating rink together with a platform around it 25 feet wide covered 37,500 square feet of ground. The area of the platform was $\frac{7}{8}$ the area of the rink. What were the dimensions of the rink?

17. The cubic contents of a produce car 33 feet long were 1848 cubic feet, while the cubic contents of a furniture car 3 feet longer, $\frac{1}{2}$ foot wider, and 1 foot higher were 2448 cubic feet. What were the dimensions of each car, if in each case the width exceeded the height?

18. Two men working together can complete a piece of work in $6\frac{2}{3}$ days. If it would take one man 3 days longer than the other to do the work alone, in how many days can each man do the work alone?

19. After a mowing machine had made the circuit of a 7-acre rectangular hay field 11 times, cutting a swath 6 feet wide each time, 4 acres of grass were still standing. How long and how wide was the field?

20. The fore wheel of a carriage makes 12 revolutions more than the hind wheel in going 240 yards. If the circumference of each wheel were 1 yard greater, the fore wheel would make 8 revolutions more than the hind wheel in going 240 yards. What is the circumference of each wheel?

21. A man loaned $\$1000$ in two unequal sums at such ates that both sums yielded the same annual interest. The urger sum at the higher rate of interest would have yielded 36 annually, the smaller sum at the lower rate, $\$16$ nnually. What sums did he invest, and at what rates of iterest?

22. A sum of money on interest for one year at a certain per ent amounted to $\$11,130$. If the rate had been $1\,\%$ less and ie principal $\$100$ more, the amount would have been the same. ind the principal and rate.

23. The town A is on a lake and 12 miles from B, which is miles from the opposite shore. A man rows across the lake d walks to B in 3 hours. In returning, he walks at the me rate as before, but rows 2 miles an hour less than before. it takes him 5 hours to return, find his rates of rowing and s rate of walking.

24. A, B, and C started together to ride a certain distance. and C rode the whole distance at uniform rates, A 2 les an hour faster than C. B rode with C for 20 miles, d then, by increasing his speed 2 miles an hour, reached s destination 40 minutes earlier than C and 20 minutes er than A. Find the distance and the rate at which each aveled.

25. The distance a body will fall in t seconds, starting from st, is given by the formula $s = \frac{1}{2}gt^2$. A man dropped a torpedo om a height and heard the report 5 seconds later. Taking $= 32.16$ and the velocity of sound 1125.6 feet per second, find, the nearest tenth of a second, the time during which the rpedo was falling.

26. A mixture of graphite and clay, to be used as "lead" in ncils, was $c\,\%$ clay and weighed p pounds. After the addition clay to make the "lead" harder, the mixture was $(c+10)\,\%$ ay and weighed 240 pounds. If graphite had been added, stead of clay, until the mixture weighed 250 pounds, the xture would have been $(c-8)\,\%$ clay. Solve for p and for c.

GRAPHIC SOLUTIONS

QUADRATIC EQUATIONS

418. Graphic solution of quadratic equations in *x*.

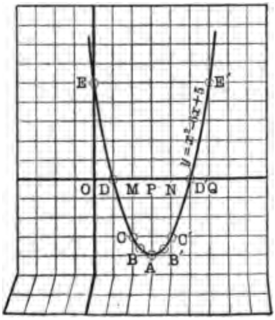
Let it be required to solve graphically, $x^2 - 6x + 5 = 0$.

To solve the equation graphically, we must first draw the graph of $x^2 - 6x + 5$. To do this, let $y = x^2 - 6x + 5$.

The graph of $y = x^2 - 6x + 5$ will represent *all* the corresponding real values of x and of $x^2 - 6x + 5$, and among them will be the values of x that make $x^2 - 6x + 5$ equal to zero, that is, the *roots* of the equation $x^2 - 6x + 5 = 0$.

In substituting values of x, when the coefficient of x^2 is $+1$, as in this instance, it is convenient to take for the first value of x a number equal to half the coefficient of x with its sign changed. Next, values of x differing from this value *by equal amounts* may be taken.

Thus, first substituting $x = 3$, it is found that $y = -4$, locating the point $A = (3, -4)$. Next give values to x differing from 3 by equal amounts, as $2\frac{1}{2}$ and $3\frac{1}{2}$, 2 and 4, 1 and 5, 0 and 6. It will be found that y has the same value for $x = 3\frac{1}{2}$ as for $x = 2\frac{1}{2}$, for $x = 4$ as for $x = 2$, etc. The table below gives a record of the points and their coördinates:



x	y	Points
3	-4	A
$2\frac{1}{2}$, $3\frac{1}{2}$	$-3\frac{3}{4}$	B, B'
2, 4	-3	C, C'
1, 5	0	D, D'
0, 6	5	E, E'

e points A; B, B'; C, C'; etc., whose coördinates are receding table, and drawing a smooth curve through them, graph of $y = x^2 - 6x + 5$ as shown in the figure.

observed that:

$= 3$, $x^2 - 6x + 5 = -4$, which is represented by the inate PA.

$= 2$ and also when $x = 4$, $x^2 - 6x + 5 = -3$, epresented by the equal *negative* ordinates MC'

$= 0$ and also when $x = 6$, $x^2 - 6x + 5 = 5$, repre- ie equal *positive* ordinates OE and QE'.

s seen that the ordinates change sign as the curve c-axis.

at D', therefore, where the ordinates are equal to 0, $x^2 - 6x + 5$ is 0, and the abscissas are $x = 1$ and

e roots of the given equation are 1 and 5.

ılf the coefficient of x with its sign changed, the number d for x, is half the sum of the roots, or their *mean value*, icient of x^2 is $+1$. This will be shown in § 433.

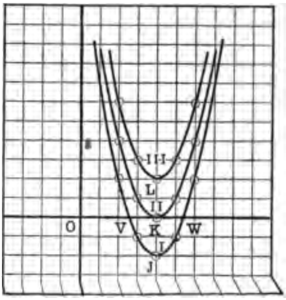
obtained by plotting the graph of $x^2 - 6x + 5$, or dratic expression of the form $ax^2 + bx + c$, is a

it be required to solve each of the equations

$+ 14 = 0,$ (1)

$+ 16 = 0,$ (2)

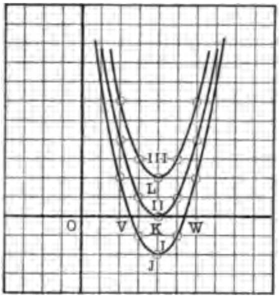
$+ 18 = 0.$ (3)



hs corresponding to (1), (2), and (3), § 418, are marked II, respectively.

of (1) are seen to be ınd $OW = 5.4$, ap-

Since graph II has only one point, K, in common with the x-axis, equation (2) appears to have only one root, $OK = 4$.



But it will be observed that if graph I, which represents two unequal real roots, OV and OW, were moved upward 2 units, it would coincide with graph II.

During this process the unequal roots of (1), OV and OW, would approach the value OK, which represents the roots of (2).

Consequently, the roots of (2) are regarded as *two* in number. They are *real* and *equal*, or *coincident*.

The movement of the graph of (1) *upward* the distance JK, or 2 units, corresponds to completing the square in (1) by *adding* 2 to each member. Since the roots of the resulting equation, $x^2 - 8x + 16 = 2$, differ from those of (2) or from the mean value $OK = 4$, by $\pm\sqrt{2}$, or $\pm\sqrt{JK}$, it is evident that the roots of (1) are represented graphically by

$$OK + \sqrt{JK} = 4 + \sqrt{2} = 5.414+,$$

and

$$OK - \sqrt{JK} = 4 - \sqrt{2} = 2.586-.$$

Since graph III has no point on the x-axis, there are no real values of x for which $x^2 - 8x + 18$ is equal to zero; that is, (3) has no real roots. Consequently, the roots are *imaginary*.

If graph III were moved *downward* 2 units, it would coincide with graph II. If the square in (3) were completed by *subtracting* 2 from each member, the roots of the resulting equation, $x^2 - 8x + 16 = -2$, would differ from the mean value by $\pm\sqrt{-2}$, or $\pm\sqrt{LK}$.

Hence, it is evident that the roots of (3) are represented graphically by

$$OK + \sqrt{LK} = 4 + \sqrt{-2},$$

and

$$OK - \sqrt{LK} = 4 - \sqrt{-2}.$$

The points J, K, and L, whose ordinates are the least *algebraically* that any points in the respective graphs can have, are *called* **minimum points.**

). When the coefficient of x^2 is $+1$, it is evident from the ding discussion that:

INCIPLES. — 1. *The roots of a quadratic in x are equal to bscissa of the minimum point, plus or minus the square root e ordinate with its sign changed.*

If the minimum point lies on the x-axis, the roots are real equal.

If the minimum point lies below the x-axis, the roots are and unequal.

If the minimum point lies above the x-axis, the roots are inary.

EXERCISES

l. Solve graphically:

$x^2 - 4x + 3 = 0.$	8. $x^2 = 6x - 10.$
$x^2 - 6x + 7 = 0.$	9. $x^2 + 4x + 2 = 0.$
$x^2 - 4x = -2.$	10. $x^2 + 3x + 4 = 0.$
$x^2 = 2(x + 1).$	11. $x^2 - 5x + 13 = 0.$
$x^2 + 2(x + 1) = 0.$	12. $x^2 - 2x + 6 = 0.$
$x^2 - 4x + 6 = 0.$	13. $x^2 - 4x - 1 = 0.$
$x^2 - 2x - 2 = 0.$	14. $x^2 + 7x + 14 = 0.$

Solve graphically $4x - 2x^2 + 1 = 0.$

GESTION. — On dividing both members of the given equation by e coefficient of x^2, the equation becomes

$$x^2 - 2x - \tfrac{1}{2} = 0.$$

roots may be found by plotting the graph of $y = x^2 - 2x - \frac{1}{2}$.

ve graphically:

$2x^2 + 8x + 7 = 0.$	18. $12x - 4x^2 - 1 = 0.$
$2x^2 - 12x + 15 = 0.$	19. $11 + 8x + 2x^2 = 0.$

TE. — Another method of solving quadratic equations graphically is *in § 426.*

422. Graphs of quadratic equations in x and y.

EXERCISES

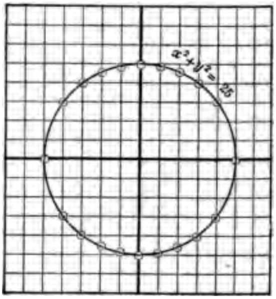
1. Construct the graph of the equation $x^2 + y^2 = 25$.

SOLUTION. — Solving for y, $y = \pm\sqrt{25 - x^2}$.

Since any value numerically greater than 5 substituted for x will n the value of y imaginary, we substitute only values of x between -5 $+5$. The corresponding values of y, or of $\pm\sqrt{25 - x^2}$, are recorded ir table below.

It will be observed that each value substituted for x, except ± 5, g two values of y, and that values of x numerically equal give the s values of y; thus, when $x = 2$, $y = \pm 4.6$, and also when $x =$ $y = \pm 4.6$.

x	y
0	± 5
± 1	± 4.9
± 2	± 4.6
± 3	± 4
± 4	± 3
± 5	0



The values given in the table serve to locate twenty points of the g of $x^2 + y^2 = 25$. Plotting these points and drawing a smooth c through them, the graph is apparently a *circle*. It may be prove geometry that this graph is a circle whose radius is 5.

The graph of any equation of the form $x^2 + y^2 = r^2$ is a whose radius is r and whose center is at the origin.

2. Construct the graph of the equation $(x - 2)^2 + (y - 3)^2$

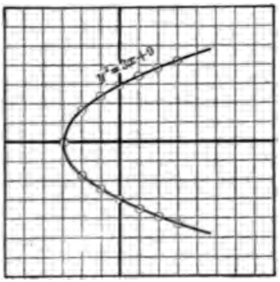
The graph of any equation of the form $(x - a)^2 + (y - b)^2 =$ is a **circle** whose radius is r and whose center is at t point (a, b).

3. Construct the graph of the equation $y^2 = 3x + 9$.

SOLUTION. — Solving for y, $y = \pm\sqrt{3x + 9}$.

It will be observed that any value smaller than -3 substituted for x l make y imaginary; consequently, no point of the graph lies to the of $x = -3$. Beginning with $x = -3$, we substitute values for x and ermine the corresponding values of y, as recorded in the table:

x	y
-3	0
-2	± 1.7
-1	± 2.4
0	± 3
1	± 3.5
2	± 3.9
3	± 4.2



lotting these points and drawing a smooth curve through them, the h obtained is apparently a *parabola* (§ 418).

he graph of any equation of the form $y^2 = ax + c$ is a bola.

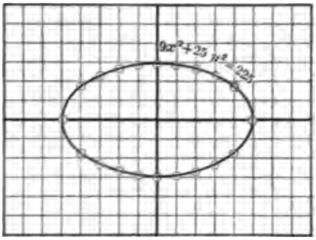
Construct the graph of the equation $9x^2 + 25y^2 = 225$.

LUTION. — Solving for y, $y = \pm \frac{1}{5}\sqrt{225 - 9x^2}$.

nce any value numerically greater than 5 substituted for x will make alue of y imaginary, no point of the graph lies farther to the right the left of the origin than 5 units; consequently, we substitute for x values between -5 and $+5$.

rresponding values of x and y are given in the table:

x	y
0	± 3
± 1	± 2.9
± 2	± 2.7
± 3	± 2.4
± 4	± 1.8
± 5	0



otting these twenty points and drawing a smooth curve through them, ave *the graph of $9x^2 + 25y^2 = 225$*, which is called an ellipse.

The graph of any equation of the form $b^2x^2 + a^2y^2 = a^2b^2$ is a **ellipse.**

5. Construct the graph of the equation $4x^2 - 9y^2 = 36$.

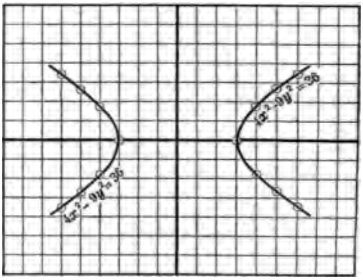
SOLUTION

Solving for y, $y = \pm \frac{1}{3}\sqrt{4x^2 - 36}.$

Since any value numerically less than 3 substituted for x will make th value of y imaginary, no point of the graph lies between $x = +3$ an $x = -3$; consequently, we substitute for x only values numerically great than 3.

Corresponding values of x and y are given in the table:

x	y
± 3	0
± 4	± 1.8
± 5	± 2.7
± 6	± 3.5



Plotting these fourteen points, it is found that half of them are on on side of the y-axis and half on the other side, and since there are no point of the curve between $x = +3$ and $x = -3$, the graph has two separate *branches*, that is, it is *discontinuous*.

Drawing a smooth curve through each group of points, the two branches thus constructed constitute the graph of the equation $4x^2 - 9y^2 = 36$, which is an *hyperbola*.

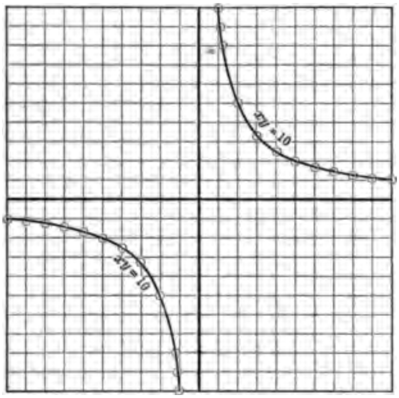
The graph of any equation of the form $b^2x^2 - a^2y^2 = a^2b^2$ is an **hyperbola.** An hyperbola has two **branches** and is called a **discontinuous** curve.

6. Construct the graph of the equation $xy = 10$.

SOLUTION

Substituting values for x and solving for y, the corresponding values ound are as given in the table on the next page.

x	y	x	y
1	10	-1	-10
2	5	-2	-5
3	$3\frac{1}{3}$	-3	$-3\frac{1}{3}$
4	$2\frac{1}{2}$	-4	$-2\frac{1}{2}$
5	2	-5	-2
6	$1\frac{2}{3}$	-6	$-1\frac{2}{3}$
7	$1\frac{3}{7}$	-7	$-1\frac{3}{7}$
8	$1\frac{1}{4}$	-8	$-1\frac{1}{4}$
9	$1\frac{1}{9}$	-9	$-1\frac{1}{9}$
10	1	-10	-1



Plotting these points and drawing a smooth curve through each group of points, the two branches of the curve found constitute the graph of the equation $xy = 10$, which is an *hyperbola*.

The graph of any equation of the form $xy = c$ is an **hyperbola**.

Construct the graph of:

7. $x^2 + y^2 = 9$.
8. $y^2 = 5x + 8$.
9. $9x^2 + 16y^2 = 144$.
10. $9x^2 - 16y^2 = 144$.
11. $xy = 12$.
12. $(x-1)^2 + (y-2)^2 = 16$.

423. Summary. — The types of equations and their respective graphs, here summarized, will aid the student in plotting graphs, but he will meet other forms of equations that will have some of the same kinds of graphs, the varieties in equations giving rise to varieties in form, size, or location of the graphs.

For example, § 422, exercises 1 and 2, are both equations of the circle, the first having its center at the origin and the second at the point (2, 3).

It is possible to determine many characteristics of the various graphs from their equations alone, but a discussion of this is beyond the province of algebra. In the study of graphs, therefore, the student will rely principally on plotting a sufficient number of points to determine their form accurately.

The following *types* have been studied:

I.	$ax + by = c$ (§ 267)	Straight line
II.	$x^2 + y^2 = r^2$	Circle
III.	$(x - a)^2 + (y - b)^2 = r^2$	Circle
IV.	$\begin{cases} ax^2 + bx + c = 0, \text{ or} \\ y = ax^2 + bx + c \end{cases}$	Parabola
V.	$y^2 = ax + c$	Parabola
VI.	$b^2x^2 + a^2y^2 = a^2b^2$	Ellipse
VII.	$b^2x^2 - a^2y^2 = a^2b^2$	Hyperbola
VIII.	$xy = c$	Hyperbola

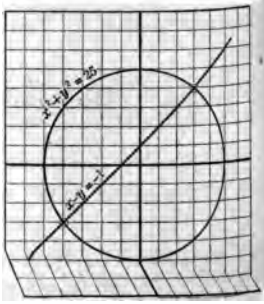
424. Graphic solution of simultaneous equations involving quadratics.

The graphic method of solving simultaneous equations that involve quadratics is precisely the same as for simultaneous linear equations (§ 271). Construct the graph of each equation, both being referred to the same axes, and determine the coördinates of the points where the graphs intersect. If they do not intersect, interpret this fact.

The student should construct the following graphs for himself. Roots are expected to the nearest tenth of a unit. To obtain this degree of accuracy, numerous points should be plotted and a scale of about $\frac{1}{2}$ inch to 1 unit should be used.

EXERCISES

425. 1. Solve graphically $\begin{cases} x^2 + y^2 = 25, \\ x - y = -1. \end{cases}$



Solution. — Constructing the graphs of these equations, we find the first, as in § 422, exercise 1, to be a circle; and the second, as in § 267, a straight line.

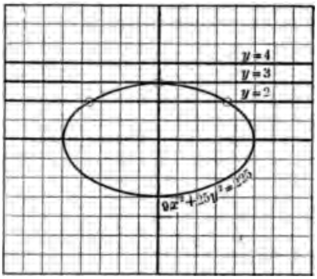
The straight line intersects the circle in two points, $(-4, -3)$ and $(3, 4)$. Hence, there are two solutions,

$$x = -4,\ y = -3;\ \text{and}\ x = 3,\ y = 4.$$

Test. — The student may test the *roots found* graphically by performing *the numerical* solution.

ve graphically $\begin{cases} 9x^2 + 25y^2 = 225, \\ y = 2. \end{cases}$

ON. — On constructing the r the first, see exercise 4, is found that they intersect its $x = 3.7$, $y = 2$, $x = -3.7$,



he graphs have these two common, and no others, their s are the only values of x and isfy both equations, and are ought.

that the pairs of values $x =$ and $x = -3.7$, $y = 2$, are lifferent, or *unequal*.

-The roots are estimated to the nearest tenth; their accuracy sted by performing the numerical solution.

ve graphically $\begin{cases} 9x^2 + 25y^2 = 225, \\ y = 3. \end{cases}$

ON. — Imagine the straight line $y = 2$ in the figure for exercise upward until it coincides with the line $y = 3$. The real unequal sented by the coördinates of the points of intersection approach nd when the line becomes the tangent line $y = 3$, they coincide. e given system of equations has *two real equal roots*, $x = 0$, $x = 0$, $y = 3$.

d the nature of the roots of $\begin{cases} 9x^2 + 25y^2 = 225, \\ y = 4. \end{cases}$

ON. — Imagine the straight line $y = 2$ in the figure for exercise upward until it coincides with the line $y = 4$. The graphs will ave any points in common, showing that the given equations *mmon real values* of x and y.

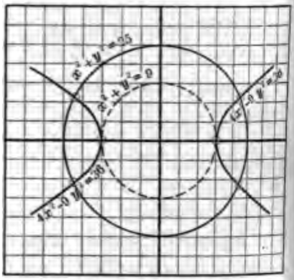
own by the numerical solution of the equations that there are and that both are *imaginary*.

m of two independent simultaneous equations in x and ple and the other quadratic, has two roots.

ots are real and unequal if the graphs intersect, real and he graphs are tangent to each other, and imaginary if the ve no points in common.

5. Solve graphically

$$\begin{cases} 4x^2 - 9y^2 = 36, \\ x^2 + y^2 = 25. \end{cases}$$



SOLUTION. — The graphs (see exercises 5 and 1, § 422) show that both of the given equations are satisfied by *four* different pairs of real values of x and y:

$$\begin{cases} x = 4.5; \quad 4.5; \quad -4.5; \quad -4.5; \\ y = 2.2; \quad -2.2; \quad -2.2; \quad 2.2. \end{cases}$$

6. What would be the nature of the roots in exercise 5, if the second equation were $x^2 + y^2 = 9$?

A system of two independent simultaneous quadratic equations in x and y has four roots.

An intersection of the graphs represents a real root, and a point of tangency, a pair of equal real roots. If there are less than four real roots, the other roots are imaginary.

Find by graphic methods, to the nearest tenth, the real roots of the following, and the number of imaginary roots, if there are any. Discuss the graphs and the roots.

7. $\begin{cases} 4x^2 - 9y^2 = 36, \\ x - 3y = 1. \end{cases}$

8. $\begin{cases} 4x^2 - 9y^2 = 36, \\ 4x^2 + 9y^2 = 36. \end{cases}$

9. $\begin{cases} 9x^2 + 16y^2 = 144, \\ x^2 + y^2 = \frac{49}{4}. \end{cases}$

10. $\begin{cases} x^2 + y^2 = 4, \\ x = y - 5. \end{cases}$

11. $\begin{cases} x^2 - 4y^2 = 4, \\ x^2 + y^2 = 4. \end{cases}$

12. $\begin{cases} x - y = 2, \\ xy = -1. \end{cases}$

13. $\begin{cases} 4x^2 - 9y^2 = 36, \\ 4y = x^2 - 16. \end{cases}$

14. $\begin{cases} 9x^2 + 16y^2 = 144, \\ 3x + 4y = 12. \end{cases}$

15. $\begin{cases} x^2 + y^2 = 9, \\ y = x^2 - 5x + 6. \end{cases}$

16. $\begin{cases} x^2 + y^2 = 9, \\ x = y^2 + 5y + 6. \end{cases}$

17. $\begin{cases} y = x^2 - 4, \\ x = (y+1)(y+4). \end{cases}$

18. $\begin{cases} y = x^2 - 5x + 4, \\ x = y^2 - 4y + 3. \end{cases}$

19. $\begin{cases} y^2+x^2+y-2x+1=0, \\ y^2+x^2+3y-4x+3=0. \end{cases}$

ot possible to solve *any* two simultaneous equations . *y*, that involve quadratics, *by quadratic methods*, ximate values of the real roots may always be found *aphic method.*

he following by both methods, if you can:

$\begin{cases} x^2+y^2=26, \\ x^2y+y=26. \end{cases}$ 21. $\begin{cases} x^2+y=7, \\ y^2+x=11. \end{cases}$

nother graphic method of solving quadratic equations 18).

been seen that the real roots of *simultaneous equa-* the coördinates of the points where their graphs inter- re tangent to each other, and that when there is no common, the roots are imaginary.

418–421, it was found that the real roots of a *quadratic* were the abscissas of the points where the graph of lratic expression crossed or touched the x-axis, and n it had no point in common with the x-axis, the roots ginary.

er words the solution of a quadratic equation in x was depend upon the solution of the simultaneous system,

$$\begin{cases} y=ax^2+bx+c, & \text{(a parabola)} \\ y=0, & \text{(a straight line)} \end{cases}$$

ld being the equation of the x-axis.

following method, by substituting y for x^2 in the given ,

$$ax^2+bx+c=0,$$

tion is divided into the simultaneous system,

$$\begin{cases} ay+bx+c=0, & \text{(a straight line)} \\ y=x^2. & \text{(a parabola)} \end{cases}$$

lution of this system for x gives the required roots of $+c=0.$

It will be observed that whether system I or II is u the solution requires the construction of a parabola an straight line, but the advantage of using II instead of I lie the fact that the parabola $y = x^2$ is the same for all quadr equations in x and when once constructed can be used for s ing any number of equations, while with I a different paral must be constructed for each equation solved.

EXERCISES

427. 1. Solve graphically the equation $x^2 - 2x - 8 = 0$.

SOLUTION

Substituting y for x^2, we have

$$y - 2x - 8 = 0.$$

Consequently, the values of x that satisfy the system,

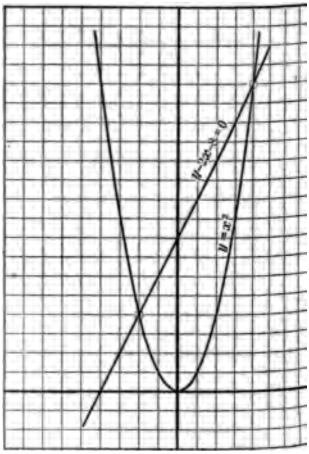
$$\begin{cases} y - 2x - 8 = 0, \\ y = x^2, \end{cases}$$

are the same as those that satisfy the given equation.

Constructing the graph of $y = x^2$, we have the parabola shown in the figure.

Constructing the graph of $y - 2x - 8 = 0$, a straight line, we find that it intersects the parabola at $x = -2$ and $x = 4$.

Hence, the roots of the equation $x^2 - 2x - 8 = 0$ are -2 and 4.



Solve graphically, giving roots to the nearest tenth:

2. $x^2 + x - 2 = 0$.
3. $x^2 - x - 6 = 0$.
4. $x^2 - 3x - 4 = 0$.
5. $x^2 - 2x - 15 = 0$.
6. $x^2 + 5x - 14 = 0$.
7. $x^2 - 7x - 18 = 0$.
8. $2x^2 - x = 6$.
9. $2x^2 - x - 15 = 0$.
10. $3x^2 + 5x - 28 = 0$.
11. $6x^2 - 7x - 20 = 0$.
12. $8x^2 + 14x - 15 = 0$.
13. $15x^2 + 2x - 20 = 0$.

ROPERTIES OF QUADRATIC EQUATIONS

28. **Nature of the roots.**

n the following discussion the student should keep in ıd the distinctions between **rational and irrational, real** and **ginary.**

For example, 2 and $\sqrt{4}$ are *rational* and also *real;* $\sqrt{2}$ and $\sqrt{5}$ are *tional,* but *real;* $\sqrt{-2}$ and $\sqrt{-5}$ are *irrational* and also *imaginary.*

29. Every quadratic equation may be reduced to the form

$$ax^2 + bx + c = 0,$$

which a is positive and b and c are positive or negative.

Denote the roots by r_1 and r_2. Then, § 390,

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

n examination of the above values of r_1 and r_2 will show t the nature of the roots, as real or imaginary, rational or tional, may be determined by observing whether $\sqrt{b^2 - 4ac}$ eal or imaginary, rational or irrational. Hence,

RINCIPLES. — In any quadratic equation, $ax^2 + bx + c = 0$, n a, b, and c represent real and rational numbers:

If $b^2 - 4ac$ *is* **positive,** *the roots are* **real** *and* **unequal.**

If $b^2 - 4ac$ *equals* **zero,** *the roots are* **real** *and* **equal.**

If $b^2 - 4ac$ *is* **negative,** *the roots are* **imaginary.**

If $b^2 - 4ac$ *is a* **perfect square** *or equals* **zero,** *the roots are* **nal;** *otherwise, they are irrational.*

0. The expression $b^2 - 4ac$ is called the **discriminant** of quadratic equation $ax^2 + bx + c = 0$.

431. If a is positive and b and c are positive or negative, th **signs of the roots** of $ax^2 + bx + c = 0$, that is, the signs of

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a},$$

may be determined from the signs of b and c.

Thus, if c is positive, $-b$ is numerically greater than $\pm\sqrt{b^2 - 4ac}$, whence both roots have the sign of $-b$; if c is negative, $-b$ is numerically less than $\pm\sqrt{b^2 - 4ac}$, whence r_1 is positive and r_2 is negative. The root having the sign opposite to that of b is the greater numerically. Hence,

Principle. — *If c is positive, both roots have the sign opposite to that of b; if c is negative, the roots have opposite signs, and the numerically greater root has the sign opposite to that of b.*

Note. — If $b = 0$, the roots have opposite signs. (See also § 378.)

EXERCISES

432. 1. What is the nature of the roots of $x^2 - 7x - 8 = 0$?

Solution. — Since $b^2 - 4ac = 49 + 32 = 81 = 9^2$, a positive number and a perfect square, by § 429, Prin. 1, the roots are real and unequal; and by Prin. 4, rational.

Since c is negative, by § 431, Prin., the roots have opposite signs and, b being negative, the positive root is the greater numerically.

2. What is the nature of the roots of $3x^2 + 5x + 3 = 0$?

Solution. — Since $b^2 - 4ac = 25 - 36 = -11$, a negative number, by § 429, Prin. 3, both roots are imaginary.

Find, without solving, the nature of the roots of:

3. $x^2 - 5x - 75 = 0$.
4. $x^2 + 5x + 6 = 0$.
5. $x^2 + 7x - 30 = 0$.
6. $x^2 - 3x + 5 = 0$.
7. $x^2 + 3x - 5 = 0$.
8. $4x^2 - 4x + 1 = 0$.
9. $4x^2 + 6x - 4 = 0$.
10. $x^2 + x + 2 = 0$.
11. $4x^2 + 16x + 7 = 0$.
12. $9x^2 + 12x + 4 = 0$.

13. For what values of m will the equation

$$2x^2 + 3mx + 2 = 0$$

re equal roots? imaginary roots?

Solution

he roots will be equal, if the discriminant equals zero (§ 429, Prin. 2);
t is, if $(3m)^2 - 4 \cdot 2 \cdot 2 = 0,$
solving, if $m = \frac{4}{3}$ or $-\frac{4}{3}.$
he roots will be imaginary, if the discriminant is negative (§ 429,
. 3); that is, if $(3m)^2 - 4 \cdot 2 \cdot 2$ is negative,
ch will be true when m is numerically less than $\frac{4}{3}$.

4. For what values of m will $9x^2 - 5mx + 25 = 0$ have al roots? real roots? imaginary roots?

5. For what values of a will the roots of the equation

$$4x^2 - 2(a-3)x + 1 = 0$$

eal and equal? real and unequal? imaginary?

6. Find the values of m for which the roots of the equation

$$4x^2 + mx + x + 1 = 0$$

equal. What are the corresponding values of x?

7. For what values of n are the roots of the equation

$$3x^2 + 1 = n(4x - 2x^2 - 1) \text{ real and equal?}$$

8. For what value of a are the roots of the equation

$$ax^2 - (a-1)x + 1 = 0$$

erically equal but opposite in sign? Find the roots for value of a.

. For what value of d has $x^2 + (2-d)x = 3d^2 - 27$ a root? Find both roots for this value of d.

. For what values of m will the roots of the equation

$$(m + \tfrac{5}{2})x^2 - 2(m+1)x + 2 = 0 \text{ be equal?}$$

. Solve the simultaneous equations for x and y

$$\begin{cases} 3x^2 - 4y^2 = 8, \\ 5(x-k) - 4y = 0. \end{cases}$$

or what values of k are the roots real? imaginary? equal?

433. Relation of roots and coefficients.

Any quadratic equation, as $ax^2 + bx + c = 0$, may l
by dividing both members by the coefficient of x^2, t
$x^2 + px + q = 0$, whose roots by actual solution are f

$$r_1 = \frac{-p + \sqrt{p^2 - 4q}}{2} \text{ and } r_2 = \frac{-p - \sqrt{p^2 -}}{2}$$

Adding the roots, $r_1 + r_2 = \frac{-2p}{2} = -p.$

Multiplying the roots, $r_1 r_2 = \frac{p^2 - (p^2 - 4q)}{4} = q.$

Hence, we have the following:

PRINCIPLE. — *The sum of the roots of a quadrati having the form $x^2 + px + q = 0$ is equal to the coeff with its sign changed, and their product is equal to tl term.*

434. Formation of quadratic equations.

Substituting $-(r_1 + r_2)$ for p, and $r_1 r_2$ for q (§ 4
equation $x^2 + px + q = 0$, we have

$$x^2 - (r_1 + r_2)x + r_1 r_2 = 0.$$

Expanding, $x^2 - r_1 x - r_2 x + r_1 r_2 = 0.$

Factoring, $(x - r_1)(x - r_2) = 0.$

Hence, to form a quadratic equation whose roots a

Subtract each root from x and place the product of t ders equal to zero.

EXERCISES

435. 1. Form an equation whose roots are -5 an

SOLUTION. $(x + 5)(x - 2) = 0$, or $x^2 + 3x - 10 = 0$.

Or, since the sum of the roots with their signs changed *or 3, and the* product of the roots is -10, (§ 433) the
$x^2 + 3x - 10 = 0$.

Form the equation whose roots are:

2. 6, 4.
3. 5, -3.
4. 3, $-\frac{1}{3}$.
5. $\frac{2}{3}$, $\frac{5}{8}$.
6. -2, $-\frac{1}{2}$.
7. $-\frac{1}{2}$, $-\frac{3}{2}$.
8. a, $-3a$.
9. $a+2$, $a-2$.
10. $b+1$, $b-1$.
11. $a+b$, $a-b$.
12. $\sqrt{a}-\sqrt{b}$, $\sqrt{b}$.
13. $\frac{1}{2}(a\pm\sqrt{b})$.
14. $3+\sqrt{2}$, $3-\sqrt{2}$.
15. $2-\sqrt{5}$, $2+\sqrt{5}$.
16. $2\pm\sqrt{3}$.
17. $-\frac{1}{2}(3\pm\sqrt{6})$.
18. $\frac{1}{2}(-1\pm\sqrt{2})$.
19. $a(2\pm2\sqrt{5})$.

20. What is the sum of the roots of $2m^2x^2-(5m-1)x=6$? For what values of m is the sum equal to 2?

21. When one of the roots of $ax^2+bx+c=0$ is twice the other, what is the relation of b^2 to a and c?

Solution

Writing $ax^2+bx+c=0$ in the form

$$x^2+\frac{b}{a}x+\frac{c}{a}=0, \tag{1}$$

and representing the roots by r and $2r$, we have

$$r+2r=3r=-\frac{b}{a}, \tag{2}$$

and

$$r\cdot 2r=2r^2=\frac{c}{a}. \tag{3}$$

On substituting the value of r obtained from (2) in (3) and reducing,

$$b^2=\tfrac{9}{2}ac.$$

22. Obtain an equation expressing the condition that one root of $4x^2-3ax+b=3$ is twice the other.

23. Find the condition that one root of $ax^2+bx+c=0$ shall be greater than the other by 3.

24. When one root of the general quadratic $ax^2+bx+c=0$ is the reciprocal of the other, what is the relation between a and c?

25. If the roots of $ax^2+bx+c=0$ are r_1 and r_2, write an equation whose roots are $-r_1$ and $-r_2$.

26. Obtain the sum of the squares of the roots $2x^2 - 12x + 3 = 0$, without solving the equation.

SOLUTION

$$\text{Sum of roots} = r_1 + r_2 = 6. \quad (1)$$

$$\text{Product of roots} = r_1 r_2 = \tfrac{3}{2}. \quad (2)$$

Squaring (1), $r_1^2 + r_2^2 + 2r_1r_2 = 36.$ (3)

(2) × 2, $2r_1r_2 = 3.$ (4)

(3) − (4), $r_1^2 + r_2^2 = 33.$

Find, without solving the equation:

27. The sum of the squares of the roots of $x^2 - 5x - 6 = 0$.

28. The sum of the cubes of the roots of $2x^2 - 3x + 1 = 0$.

29. The difference between the roots of $12x^2 + x - 1 = 0$.

30. The square root of the sum of the squares of the roots of $x^2 - 7x + 12 = 0$.

31. The sum of the reciprocals of the roots of $ax^2 + bx + c = 0$.

SUGGESTION. $\dfrac{1}{r_1} + \dfrac{1}{r_2} = \dfrac{r_1 + r_2}{r_1 r_2}.$

32. The difference between the reciprocals of the roots of $8x^2 - 10x + 3 = 0$.

436. The number of roots of a quadratic equation.

It has been seen (§ 433) that any quadratic equation may be reduced to the form $x^2 + px + q = 0$, which has *two* roots, as r_1 and r_2. To show that the equation cannot have more than two roots, write it in the form given in § 434, namely,

$$(x - r_1)(x - r_2) = 0. \quad (1)$$

If the equation has a third root, suppose it is r_3.

Substituting r_3 for x in (1), we have

$$(r_3 - r_1)(r_3 - r_2) = 0,$$

which is impossible, if r differs from both r_1 and r_2. Hence,

PRINCIPLE. — *A quadratic equation has two and only two roots.*

. **Factoring by completing the square.**

e method of factoring is useful in solving quadratic equa- when the factors are rational and readily seen. In more ılt cases we complete the square. This more powerful ɔd is useful also in factoring quadratic expressions the ꞉s of which are irrational or otherwise difficult to obtain.

EXERCISES

ʿ. 1. Factor $2x^2 + 5x - 3$.

SOLUTION

$$2x^2 + 5x - 3 = 0.$$

iding by 2, etc., $x^2 + \frac{5}{2}x = \frac{3}{2}$.

ıpleting the square, $x^2 + \frac{5}{2}x + \frac{25}{16} = \frac{49}{16}$.

'ing, $x = \frac{1}{2}$ or -3.

ming an equation having these roots, § 434,

$$(x - \tfrac{1}{2})(x + 3) = 0.$$

tiplying by 2 because we divided by 2,

$$(2x - 1)(x + 3) = 2x^2 + 5x - 3 = 0.$$

ıce, the factors of $2x^2 + 5x - 3$ are $2x - 1$ and $x + 3$.

ctor:

$4x^2 - 4x - 3$.	6. $7x^2 + 13x - 2$.
$5x^2 + 3x - 2$.	7. $24x^2 - 10x - 25$.
$3x^2 + 14x - 5$.	8. $10x^2 + 21x - 10$.
$8x^2 - 14x + 3$.	9. $15x^2 - 5.5x - 1$.

Factor $x^2 + 2x - 4$.

SOLUTION

$$x^2 + 2x - 4 = 0.$$

ıpleting the square, $x^2 + 2x + 1 = 5$.

ring, $x = -1 + \sqrt{5}$ or $-1 - \sqrt{5}$.

ıce, § 434, $(x + 1 - \sqrt{5})(x + 1 + \sqrt{5}) = x^2 + 2x - 4 = 0$.

t is, the factors of $x^2 + 2x - 4$ are $x + 1 - \sqrt{5}$ and $x + 1 + \sqrt{5}$.

Factor:

11. $x^2+4x-6.$

12. $y^2-6y+3.$

13. $z^2-5z-1.$

14. $x^2+x+1.$

15. $a^2+3a-5.$

16. $t^2+3t+7.$

17. Factor $2-3x-2x^2.$

SUGGESTION. — Since $2-3x-2x^2=-2(x^2+\frac{3}{2}x-1)$, factor $x^2+\frac{3}{2}x-1$, in which the coefficient of x^2 is $+1$, and multiply the result by -2.

18. $2x^2+2x-1.$

19. $9x^2-4x+1.$

20. $24x-16x^2-3.$

21. $9a^2-12a+5.$

22. $16v(1-v)-9.$

23. $16(3+n)+3n^2.$

24. Factor $100x^2+70xy-119y^2.$

SUGGESTION. — The coefficient of x^2 being a perfect square, complete the square directly; do not divide by 100.

25. $4b^2-48b+143.$

26. $9r^2-12r+437.$

27. $4a^2+12a-135.$

28. $16p(p+1)-1517.$

29. $25e^2-2h(5e-2h).$

30. $3h(4k-3h)-7k^2.$

31. Factor $x^4+4x^3+8x^2+8x-5.$

SOLUTION

Let $$x^4+4x^3+8x^2+8x-5=0.$$

Completing the square,

$$(x^4+4x^3+4x^2)+4(x^2+2x)+4=9.$$

Extracting the square root, $x^2+2x+2=3$ or -3.

$$\therefore x^4+4x^3+8x^2+8x-5=(x^2+2x+2-3)(x^2+2x+2+3)$$
$$=(x^2+2x-1)(x^2+2x+5).$$

Factor the following polynomials:

32. $x^4+6x^3+11x^2+6x-8.$

33. $x^6+2x^5+5x^4+8x^3+8x^2+8x+3.$

34. $x^6-4x^5+6x^4+6x^3-19x^2+10x+9.$

35. $4x^6+12x^5+25x^4+40x^3+40x^2+32x+15.$

Resolve $x^4 + 1$ into factors of the second degree.

SOLUTION

$$x^4 + 1 = x^4 + 2x^2 + 1 - 2x^2$$
$$= (x^2 + 1)^2 - (x\sqrt{2})^2$$
$$= (x^2 + x\sqrt{2} + 1)(x^2 - x\sqrt{2} + 1).$$

:. — Each of these quadratic factors may be resolved into two of the first degree by completing the square. The factors are: $\sqrt{2} + \frac{1}{2}\sqrt{-2})$, $(x + \frac{1}{2}\sqrt{2} - \frac{1}{2}\sqrt{-2})$, $(x - \frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{-2})$, $- \frac{1}{2}\sqrt{2} - \frac{1}{2}\sqrt{-2})$.

lve into quadratic factors:

$x^4 + 16.$ 39. $x^4 + 2a^2x^2 + 4a^4.$

$x^4 + b^4.$ 40. $v^4 - 4nv - 2n^4.$

Values of a quadratic expression.

expression that has different values corresponding to it values of x is called a **function** of x.

is a function of x, for when $x = 1, 2, 3, \cdots$, $x^2 - 2x = -1, 0, 3, \cdots$.

ie following discussions only *real* values of x are con-.

EXERCISES

1. What values has the function $x^2 - 2x - 3$ corre-ig to very large positive or negative values of x?

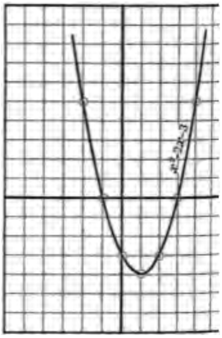
ssion. — When x is very large and either positive or negative, e of $x^2 - 2x - 3$ is approximately equal to that of its largest term, is when $x = \pm 100$, $x^2 - 2x - 3 = 10{,}000$, approximately; when 000, $x^2 - 2x - 3 = 1{,}000{,}000$, approximately.

x^2 is always positive, whether x is positive or negative, for very lues of x, $x^2 - 2x - 3$ is very large, and positive; and by making nough we can make $x^2 - 2x - 3$ greater than any number that assigned, however great.

umber that may become greater than any assignable r is called an **infinite number**.

symbol of an infinite number is ∞, read 'infinity.'

2. Interpret the conclusion of exercise 1 graphically.



DISCUSSION. — Draw the graph of $x^2 - 2x - 3$, plotting values of x as abscissas and values of $x^2 - 2x - 3$ as ordinates (§ 418).

In the discussion of exercise 1, it is seen that when $x = -\infty$, and also when $x = +\infty$, $x^2 - 2x - 3 = +\infty$; that is, when x increases without limit, either in the negative direction, or in the positive direction along the x-axis, $x^2 - 2x - 3$, represented by ordinates to the curve, increases without limit in the positive direction.

Referring to the graph of $x^2 - 2x - 3$ and observing form of the function itself, a brief discussion for real va of x may be given as follows:

(*a*) As x increases continuously from $-\infty$ to $+1$, $x^2 - 2$: decreases continuously from $+\infty$ to its minimum value, crossing the x-axis at $x = -1$, which is therefore a root of equation $x^2 - 2x - 3 = 0$.

(*b*) As x increases continuously from $+1$ to $+\infty$, $x^2 - 2$ increases continuously from its minimum value, -4, to - crossing the x-axis at $x = 3$, which is therefore the other of the equation $x^2 - 2x - 3 = 0$.

(*c*) The function is positive for all values of x outside limits $x = -1$ and $x = 3$, and negative for all values within these limits.

When the coefficient of x^2 is $+1$, the abscissa of the minimum po half the coefficient of x with its sign changed (§ 418).

In a similar way discuss the following functions:

3. $x^2 - 5x + 6$.

4. $x^2 - 2x - 8$.

5. $x^2 + 2x - 15$.

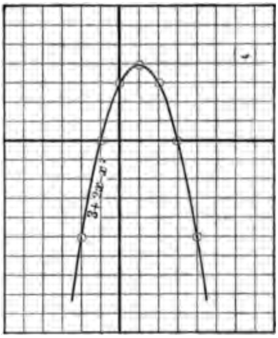
6. $x^2 + 5x + 4$.

7. $x^2 - 9$.

8. $x^2 + x + 1$.

the maximum value of $3+2x-x^2$.

IRST SOLUTION



$x-x^2=-(x^2-2x-3)$, and
is a minimum value at $x=1$
the given function has a
le at $x=1$.
, $3+2x-x^2=4$, the *max-*

:COND SOLUTION

$-x^2=y$.
x, $x=1\pm\sqrt{4-y}$.
ist be *real*, $4-y=0$ or a
er.
, $y=4$.
positive number, y is *less* than 4.
4 is the *maximum* value of the function.

lete the discussion of the values of $3+2x-x^2$.

e values of the following:

$-x^2$. 14. $2x^2+5x-3$.

$x-x^2$. 15. $2x^2+3x+2$.

$16x+15$. 16. $4x-6-x^2$.

hat values of x is x^2-5x+6 positive?

SOLUTION

$$x^2-5x+6=(x-2)(x-3).$$

is positive when both factors are positive or when both are
is, when x is less than 2 or greater than 3, these values
s of the equation $x^2-5x+6=0$.

what values of x is $x^2-3x-28$ positive? nega-

that $x^2-6x+12$ is positive for all real values of
the nature of the roots of $x^2-6x+12=0$?

that $x-x^2-1$ is negative for all real values of x.

is the condition that ax^2+bx+c shall have the
r *all real* values of x?

GENERAL REVIEW

441. 1. Define power; root; like terms; transpositio simultaneous equations; surd.

2. Distinguish between known and unknown numbers.

3. Why is the sign of multiplication usually omitted b tween letters, and never omitted between figures?

4. How is multiplication like addition? division like su traction? What two meanings has the minus sign in algebra

5. When $\times$, $\div$, or both occur in connection with $+$, $-$, both in an expression, what is the sequence of operations?

6. State the law of exponents for multiplication; for div sion.

7. When is $x^n - y^n$ divisible by both $x + y$ and $x - y$?

8. When is a trinomial a perfect square? When is a fra tion in its lowest terms? What are similar fractions?

9. What operation is indicated by the radical sign? I what other way may this operation be indicated?

10. When is an expression both integral and rational? Whe are expressions said to be prime to each other?

11. By what principle may cancellation be used in reducin fractions to lowest terms?

12. During 12 hours of a certain day, the following temper tures were recorded at Helena, Montana: $-9°$, $-8°$, -8 -9, $-9°$, $-9°$, $-8°$, $+12°$, $+25°$, $+40°$, $+20°$, $+16$ nd the average temperature for the 12 hours.

13. Define the terms conditional equation; identical equation.

14. Explain the meaning of a negative integral exponent; of fractional exponent.

15. Define evolution; radical; entire surd; binomial surd; nilar surds.

16. Express the following without parentheses:

$$(a^2x^m)^n,\ -[-(a^2)^2]^2,\ (a^3)^4,\ (a^4)^3.$$

17. What is meant by the order of a surd? Illustrate your swer by giving surds of different orders.

18. Tell how to rationalize a binomial quadratic surd.

19. What powers of $\sqrt{-1}$ are real? imaginary?

20. What roots should be associated when the roots of a tem of equations are given thus: $x = \pm 2$, $y = \mp 3$?

21. Illustrate how a root may be introduced in the solution an equation; how a root may be removed.

22. Why is it specially important to test the values of the nown number found in the solution of radical equations?

3. Upon what axiom is clearing equations of fractions ed? What precautions should be taken to prevent introing roots? If roots are introduced, how may they be ected?

4. Define symmetrical equation; quadratic surd; coördinate s; imaginary number; axiom; coefficient; homogeneous nomial; elimination.

5. Explain how, in the solution of problems, negative roots uadratic equations, while mathematically correct, are often lmissible.

3. Define negative number, subtraction, and multiplication, show, *from your definition*, how the following rules may be ıced:

) "Change the sign of the subtrahend and proceed as in ition;"

) "Give the product the positive or the negative sign acing as the two factors have like or unlike signs."

27. What is a pure quadratic equation? a complete quadratic equation? Give the general form of each.

28. State two methods of completing the square in the solution of affected quadratic equations.

29. What is the relation between the factors of an expression and the roots of the equation that may be formed by putting the expression equal to 0?

30. Outline the method of solving quadratic equations by factoring.

31. Write the roots of the quadratic equation $ax^2 + bx + c = 0$. Write the discriminant of the equation. What relation between the coefficients indicates that the roots are imaginary? reciprocals of each other?

32. What is the advantage of letting $x^2 = y$ in the graphic solution of quadratic equations of the form $ax^2 + bx + c = 0$?

33. How does the graph of a quadratic equation show the fact, if the roots are real and equal? real and unequal? imaginary?

34 Prove that a quadratic equation has two and only two roots

35. Tell how to form a quadratic equation when its roots are given. Form the equation whose roots are $\frac{2}{3}$ and $\frac{1}{2}$.

36. What is the meaning of "function of x"? "infinite number"?

37. Tell how the signs of the roots of a quadratic equation may be determined without solving the equation.

38. Derive the value of the sum of the roots of the equation $x^2 + px + q = 0$; the value of the product of the roots.

39. In clearing a fractional equation of its denominators, why should we multiply by their lowest common multiple? Illustrate by showing what happens when the equation

$$\frac{2x}{x-1} - \frac{10}{x^2-1} = \frac{7}{x+1}$$

is multiplied by the product of all the denominators.

EXERCISES

42. 1. Add $x\sqrt{y}+y\sqrt{x}+\sqrt{xy}$, $x^{\frac{1}{2}}y^{\frac{1}{2}}-\sqrt{x^2y}-\sqrt{xy^2}$, $\sqrt{x^2y}$ $\sqrt{xy^2}-\sqrt{xy}$, and $y\sqrt{x}-x\sqrt{4\,y}-\sqrt{9\,xy}$.

2. What number must be subtracted from $a-b$ to give $a+c$?

3. Simplify $a-\{b-a-[a-b-(2\,a+b)+(2\,a-b)-a]-b\}$.

4. Multiply $x\sqrt{x}+x\sqrt{y}+y\sqrt{x}+y\sqrt{y}$ by $\sqrt{x}-\sqrt{y}$.

5. Multiply $2\,x^{\frac{a}{2b}}-5\,y^{\frac{a+b}{2}}$ by $2\,x^{\frac{a}{2b}}+5\,y^{\frac{a+b}{2}}$.

6. Expand $(x^n-y^n)(x^n+y^n)(x^{2n}+y^{2n})$.

7. Divide x^4-y^4 by $x-y$ by inspection. Test.

8. Divide $x^5-3\,x^2-20$ by $x-2$, by detached coefficients.

9. Show by the factor theorem that $x^{50}-b^{50}$ is divisible by b.

10. Divide $(a+b)+x$ by $(a+b)^{\frac{1}{3}}+x^{\frac{1}{3}}$.

11. Factor $9\,x^2-12\,x+4$; $9\,x^2+9\,x+2$; $x^3-3\,x+2$; a^4+1.

12. Show by the factor theorem that $x-a$ is a factor of $3\,ax^{n-1}-4\,a^n$.

13. Separate $a^{12}-1$ into six rational factors.

14. Factor $4\,(ad+bc)^2-(a^2-b^2-c^2+d^2)^2$.

15. Find the H. C. F. of $3\,x^2-x-2$ and $6\,x^2+x-2$.

16. Find the H. C. F. of $4\,x^4-11\,x^2+11\,x-12$,
$2\,x^4+x^3-4\,x^2+7\,x-15$, and $2\,x^4+x^3-x-12$.

17. Find the L. C. M. of $4\,a^2bx$, $6\,ab^2y^2$, and $2\,axy$.

18. Find the L. C. M. of x^2-y^2, $x+y$, and $xy-y^2$.

19. Reduce $\dfrac{a^2-b^2-c^2-2\,bc}{a^2-b^2+c^2+2\,ac}$ to its lowest terms.

20. Simplify $\dfrac{x}{x+1}-\dfrac{x}{1-x}+\dfrac{x^2}{x^2-1}$.

21. Simplify $\dfrac{x+y}{2x-2y}+\dfrac{x-y}{2x+2y}+\dfrac{4x^2y^2}{y^4-x^4}$.

22. Simplify $\dfrac{1}{(a-b)(b-c)}-\dfrac{1}{(c-b)(c-a)}+\dfrac{1}{(c-a)(a-b)}$.

23. Simplify $\left\{\dfrac{x}{1+\dfrac{1}{x}}+1-\dfrac{1}{x+1}\right\}\div\left\{\dfrac{x}{1-\dfrac{1}{x}}-x-\dfrac{1}{x-1}\right\}$.

24. Simplify $\dfrac{1}{x-\dfrac{1}{x+\dfrac{1}{x}}}-\dfrac{1}{x+\dfrac{1}{x-\dfrac{1}{x}}}$.

25. Raise $a-b$ to the seventh power.

26. Expand $(2a+3b)^4$; $(\sqrt{x}+\sqrt[3]{y})^6$; $(-1-\sqrt{-3})^3$.

27. Find the sixth root of 4826809.

28. Reduce $\sqrt{\frac{2}{3}}$ to its simplest form.

29. Reduce $\sqrt[4]{25a^4}$ to its simplest form.

30. Find the value of $\dfrac{1}{\sqrt{2}}$ to 3 decimal places.

31. Multiply $2+\sqrt{8}$ by $1-\sqrt{2}$; $2+\sqrt{-8}$ by $1-\sqrt{-2}$

32. Simplify $\dfrac{\sqrt{3}+3\sqrt{2}}{\sqrt{6}+2}$.

33. Show that $(ax)^0=1$.

34. Show that $ax^{-5}=\dfrac{a}{x^5}$.

35. Show that $x^{\frac{2}{3}}=\sqrt[3]{x^2}$; also that $x^{\frac{2}{3}}=(\sqrt[3]{x})^2$.

36. *Find* the value of $125^{\frac{2}{3}}$; of $\left(\dfrac{x^5}{32}\right)^{-\frac{3}{5}}$, when $x=.5$.

olve the following equations for x:

37. $\dfrac{1}{a-b}+\dfrac{a-b}{x}=\dfrac{1}{a+b}+\dfrac{a+b}{x}.$

38. $\dfrac{x-\frac{1}{a}}{c}+\dfrac{x-\frac{1}{b}}{a}+\dfrac{x-\frac{1}{c}}{b}=0.$

). $mx^2-nx=mn.$

). $x^4+\dfrac{1}{2}=\dfrac{3x^2}{2}.$

. $x^6+8=9x^3.$

42. $\sqrt{x-9}=\sqrt{x}-1.$

43. $x^2+\sqrt{x^2+16}=14.$

44. $\left(\dfrac{4}{x}+x\right)^2-\left(\dfrac{4}{x}+x\right)=20.$

45. $(1+x)^5+(1-x)^5=242.$

46. $x+x^2+(1+x+x^2)^2=55.$

47. $\dfrac{1+x}{1+x+\sqrt{1+x^2}}=a-\dfrac{1+x}{1-x+\sqrt{1+x^2}}.$

lve for x, y, and z:

. $\begin{cases}\dfrac{1}{x}+\dfrac{1}{y}=10,\\[2mm] \dfrac{3}{x}+\dfrac{2}{y}=10.\end{cases}$

). $\begin{cases}2x+3y+z=9,\\ x+2y+3z=13,\\ 3x+y+2z=11.\end{cases}$

). $\begin{cases}ax+y+z=2(a+1),\\ x+ay+z=3a+1,\\ x+y+az=a^2+3.\end{cases}$

. $\begin{cases}x^2+xy=24,\\ y^2+xy=12.\end{cases}$

. $\begin{cases}x^2+3xy=7,\\ xy+4y^2=18.\end{cases}$

53. $\begin{cases}x^2+x=26-y^2-y,\\ xy=8.\end{cases}$

54. $\begin{cases}\sqrt{xy}=12,\\ x+y-\sqrt{x+y}=20.\end{cases}$

55. $\begin{cases}xy-xy^2=-6,\\ x-xy^3=9.\end{cases}$

56. $\begin{cases}xy=x+y,\\ x^2+y^2=8.\end{cases}$

57. $\begin{cases}x^2y^2-4xy=5,\\ x^2+4y^2=29.\end{cases}$

58. $\begin{cases}2x^3+2y^3=9xy,\\ x+y=3.\end{cases}$

59. $\begin{cases}x^{\frac{1}{2}}+y^{\frac{1}{3}}=4,\\ x^{\frac{3}{2}}+y=16.\end{cases}$

Problems

443. 1. How far down a river whose current runs 3 miles an hour can a steamboat go and return in 8 hours, if its rate of sailing in still water is 12 miles an hour?

2. A woman on being asked how much she paid for her eggs, replied, "Three dozen cost as many cents as I can buy eggs for 64 cents." What was the price per dozen?

3. A man had not room in his stable for 8 of his horses, so he built an additional stable $\frac{1}{2}$ the size of the other, and then had room for 8 horses more than he had. How many horses had he?

4. In a mass of copper, lead, and tin, the copper was 5 pounds less than half the whole in weight, and the lead and tin each 5 pounds more than $\frac{1}{3}$ of the remainder. Find the weight of each.

5. A person who can walk n miles an hour has a hours at his disposal. How far may he ride in a coach that travels m miles an hour and return on foot within the allotted time?

6. A merchant sold half a car load more than half his grain; then he sold half a car load more than half the remainder, and then found that if he should sell half a car load more than half of what he still had, he would have none left. How many car loads of grain had he?

7. A man received \$2.50 per day for every day he worked, and forfeited \$1.50 for every day he was idle. If he worked 3 times as many days as he was idle and received \$24, how many days did he work?

8. A jeweler has two silver cups, and a cover worth \$1.50. The first cup with the cover on it is worth $1\frac{7}{8}$ times as much as the second cup, and the second cup with the cover on it is worth $\frac{11}{12}$ as much as the first cup. Find the value of each cup.

9. Find two numbers such that their sum, their product, and the difference of their squares are all equal.

10. A woman has 13 square feet to add to the area of e rug she is weaving. She therefore increases the length $\frac{1}{8}$ d the width $\frac{1}{4}$, which makes the perimeter 4 feet greater. nd the dimensions of the finished rug.

11. Twenty-eight tons of goods are to be transported in carts d wagons, and it is found that it will require 15 carts and 12 ıgons, or else 24 carts and 8 wagons. How much can each rt and each wagon carry?

12. There is a number whose three digits are the same. If imes the sum of the digits is subtracted from the number, remainder is 180. What is the number?

13. A and B can do a piece of work in m days, B and C in lays, A and C in p days. In what time can all together do How long will it take each alone to do it?

14. Two passengers together have 400 pounds of baggage d are charged, for the excess above the weight allowed free, cents and 60 cents, respectively. If the baggage had be-ıged to one of them, he would have been charged $1.50. ow much baggage is one passenger allowed without charge?

15. It took a number of men as many days to pave a side-ılk as there were men. Had there been 3 men more, the work ıuld have been done in 4 days. How many men were there?

16. A merchant bought two lots of tea, paying for both $34. ıe lot was 20 pounds heavier than the other, and the number cents paid per pound was in each case equal to the number pounds bought. How many pounds of each did he buy?

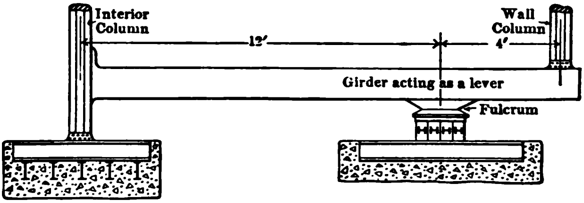
17. A and B hired a pasture for a certain sum per week. A t in 4 horses, and B as many as cost him 18 shillings per ek. Later B put in 2 additional horses, and had to pay 20 illings per week. Find the cost of the pasture per week.

18. By lowering the selling price of apples 1 cent a dozen, nan finds that he can sell 60 more than he used to sell for cents. At what price per dozen did he sell them at first?

19. A railway train, after traveling 2 hours at its usual ra was detained 1 hour by an accident. It then proceeded at $\frac{3}{5}$ its former rate, and arrived $7\frac{2}{3}$ hours behind time. If the ac dent had occurred 50 miles farther on, the train would ha arrived $6\frac{1}{3}$ hours behind time. What was the whole distai traveled by the train?

20. A and B left Chicago and walked in the same directi at uniform rates, B starting 2 hours after A and overtaki him at the 30th milestone. Had each traveled half a mile m per hour, B would have overtaken A at the 42d milestone. what rate did each travel?

21. The load on a wall column for an office building 360,000 pounds, including the weight of the column itself, a



is balanced, as shown in the figure, by a part of the load on interior column. Neglecting the weight of the girder, find t load on the fulcrum.

22. A projectile fired from a battleship was heard by t gunner to strike a mark 3360 feet away $4\frac{1}{3}$ seconds after it w fired. An officer on another vessel 5600 feet from the first ai 2240 feet from the mark heard the shot strike $1\frac{2}{3}$ secon before the report reached him. Find the velocity of the sou and the average velocity of the projectile.

23. The velocity acquired or lost by a body acted upon l gravity is given by the formula $v = gt$ (take $g = 32.16$). If bullet is fired vertically upward with an initial velocity of 20 feet per second, in how many seconds will it return to tl earth (neglecting the friction of the air)?

Using the formula $s = \frac{1}{2}gt^2$, find how high the bullet will ris

444. The following are from recent college entrance examination papers:

1. Determine graphically the roots of $4x + 5y = 24$, $3x - 2y = -5$. Give the construction in full.

2. Solve $\dfrac{3}{2x+3} + \dfrac{1}{x-5} - \dfrac{8}{2x^2 - 7x - 15} = 0.$

3. Solve for x and y, $(x-y)^2 = c^2$, $(y-a)(x-b) = 0$.

4. Find x from the equations,

$$\begin{cases} x^2 + xy + z = 2, \\ x + 2y + z = 3, \\ x - y + z = 0. \end{cases}$$

SUGGESTION. — From the second equation subtract the third.

5. Solve $\begin{cases} 2(x+y)^2 - (x+y)(x-2y) = 70, \\ 2(x+y) - 3(x-2y) = 2. \end{cases}$

6. Solve for x, y, and z,

$$\begin{cases} x + y = xy, \\ 2x + 2z = xz, \\ 3z + 3y = yz. \end{cases}$$

SUGGESTION. — Find y in terms of x, and z in terms of x; substitute these results in the remaining equation.

7. Solve $x - y - \sqrt{x-y} = 2$, $x^3 - y^3 = 2044$.

8. Solve for x and y $\begin{cases} (a+c)x - (a-c)y = 2ab, \\ (a+b)x - (a-b)y = 2ac. \end{cases}$

9. Factor $4x^4 + 1$; $27x^2 + 3x - 2$; $4x^4 + y^4 - 5x^2y^2$.

10. Resolve into prime factors:

$3(a-1)^3 - (1+a)$; $a^4 - a^2b^2 - b^2 - 1$; $3x^{-\frac{1}{2}} + 7x^{-\frac{1}{4}} - 6$.

11. Solve the equation $4x^2 + mx + 5 = 0$. For what values of m are the roots imaginary?

12. How much water must be added to 80 pounds of a 5% salt solution to obtain a 4% solution?

13. Construct the graph of the function $x^2 - 2x + 1$.

14. Under what conditions will the roots of $ax^2 + bx + c = 0$ be positive? negative? one positive and the other negative?

15. Find to four terms the square root of $x^2 - 3x + 1$.

16. Find the square root of

$$\frac{9a^2c^{2m}}{4b^{12}} - \frac{3ac^{m+n}}{b^3} + b^6c^{2n} - \frac{2^8ac^m}{b^6} + \frac{2^9b^3c^n}{3} + \frac{2^{16}}{9}.$$

17. Solve $x^2 + 7x - 3 = \sqrt{2x^2 + 14x + 2}$.

18. Simplify $\dfrac{bc(b-c) + ac(c-a) + ab(a-b)}{a^2 + bc - ac - ab}$.

19. Solve $\begin{cases} \dfrac{2x}{3} - \dfrac{5y}{12} - \left(\dfrac{3x}{2} - \dfrac{4y}{3}\right) = -\dfrac{2}{3}, \\ \dfrac{x-y}{x+y} = \dfrac{1}{5}. \end{cases}$

20. Solve $\begin{cases} \sqrt{x} - \sqrt{y} = 2, \\ (\sqrt{x} - \sqrt{y})\sqrt{xy} = 30. \end{cases}$

21. Show by the factor theorem that $a^n + b^n$ is exactly divisible by $a + b$ for all positive odd integral values of n.

22. The area of the floor of a room is 120 square feet; of one end wall, 80 square feet; and of one side wall, 96 square feet. Find the dimensions of the room.

23. Show that $\sqrt[3]{3}$ is greater than $\sqrt[5]{6}$.

24. Draw the graphs of the two equations $\begin{cases} x^2 + y^2 = 4, \\ 5x + 4y = 20, \end{cases}$ and tell the algebraic meaning of the fact that the two graphs do not intersect.

25. A rectangular piece of tin is 4 inches longer than it is wide. An open box containing 840 cubic inches is made from it by cutting a 6-inch square from each corner and turning up the ends and sides. What are the dimensions of the box?

INEQUALITIES

5. Any problem thus far presented has been such that onditions could be stated by means of one or more *equa-* . In some problems and exercises, however, the condi- are such as to lead to a *statement that one number is er or less than another.* It is the purpose of this chapter scuss such statements, for they often yield all the data ssary to the required solution.

6. One number is said to be **greater than** another when the inder obtained by subtracting the second from the first is *ive,* and to be **less than** another when the remainder ned by subtracting the second from the first is *negative.*

$a - b$ is a positive number, a is *greater* than b; but if $a - b$ is a ive number, a is *less* than b.

ly negative number is regarded as less than 0; and, of two tive numbers, that more remote from 0 is the less.

us, -1 is less than 0; -2 is less than -1; -3 is less than -2; etc.

ı algebraic expression indicating that one number is greater ss than another is called an **inequality**.

7. The **sign of inequality** is $>$ or $<$.

is placed between two unequal numbers with the opening rd the greater.

is greater than b' is written $a > b$; '*a is less than b*' is written '.

ıe expressions on the left and right, respectively, of the of inequality are called the **first** and the **second members of** *inequality.*

448. The signs $\not>$ and $\not<$ are read, respectively, '*is* *greater than*' and '*is not less than.*'

449. When the first members of two inequalities are ea greater or each less than the corresponding second membe the inequalities are said to **subsist in the same sense.**

When the first member is greater in one inequality and le in another, the inequalities are said to **subsist in a contra sense.**

$x > a$ and $y > b$ subsist in the same sense, also $x < 3$ and $y < 4$; $x > b$ and $y < a$ subsist in a contrary sense.

450. The following illustrate operations with inequalitie

1. Given $8 > 5$
   Add, $\quad 2 \quad 2$
   $8 + 2 > 5 + 2$
   That is, $10 > 7$.

2. Given $8 > 5$
   Subtract, $\quad 2 \quad 2$
   $8 - 2 > 5 - 2$
   That is, $6 > 3$.

3. Given $8 > 5$
   Multiply, $\quad 2 \quad 2$
   $8 \cdot 2 > 5 \cdot 2$
   That is, $16 > 10$.

4. Given $16 > 10$
   Divide, $\quad 2 \quad 2$
   $16 \div 2 > 10 \div 2$
   That is, $8 > 5$.

5. Given $8 > 5$
   Multiply, $\quad -2 \quad -2$
   $8 \cdot -2 < 5 \cdot -2$
   That is, $-16 < -10$.

6. Given $16 > 10$
   Divide, $\quad -2 \quad -2$
   $16 \div -2 < 10 \div -$
   That is, $-8 < -5$.

451. Principle 1. — *If the same number or equal numl be added to or subtracted from both members of an inequal the resulting inequality will subsist in the same sense.*

For, let $a > b$, and let c be any positive or negative number.

Then, § 446, $a - b = p$, a positive number.

Adding $c - c = 0$, Ax. 1, $\quad a + c - (b + c) = p$.

Therefore, $\quad a + c > b + c$.

Note. — Letters used in this chapter stand for real numbers.

452. Principle 2. — *If both members of an inequality are multiplied or divided by the same number, the resulting inequality will subsist in the same sense if the multiplier or divisor is positive, but in the contrary sense if the multiplier or divisor is negative.*

For, let $a > b.$

Then, § 446, $a - b = p$, a positive number.

Multiplying by m, $ma - mb = mp.$

If m is positive, mp is positive,

and therefore, § 446, $ma > mb.$

If m is negative, mp is negative,

and therefore, § 446, $ma < mb.$

Putting $\frac{1}{m}$ for m, the principle holds also for division.

453. Principle 3. — *A term may be transposed from one member of an inequality to the other, provided its sign is changed.*

For, let $a - b > c - d.$

Adding b to each side, Prin. 1, $a > b + c - d.$

Adding $-c$ to each side, Prin. 1, $a - c > b - d.$

454. Principle 4. — *If the signs of all the terms of an inequality are changed, the resulting inequality will subsist in the contrary sense.*

For, let $a - b > c - d.$

Multiplying each side by -1, Prin. 2,

$$-a + b < -c + d.$$

455. Principle 5. — *If the corresponding members of any number of inequalities subsisting in the same sense are added, the resulting inequality will subsist in the same sense.*

For, let $a > b,\ c > d,\ e > f$, etc.

Then, § 446, $a - b,\ c - d,\ e - f$, etc., are positive.

Hence, their sum, $a + c + e + \cdots - (b + d + f + \cdots)$, is positive;

that is, $a + c + e + \cdots > b + d + f + \cdots.$

Note.—The student should bear in mind that the *difference* of two inequalities subsisting in the *same sense*, or the *sum* of two inequalities subsisting in a *contrary sense*, may have its *first* member *greater than*, *equal to*, or *less than* its *second*.

Thus, take the inequality $12 > 6$.

Subtracting $7 > 3$, or adding $-7 < -3$, the result is $5 > 3$.

Subtracting $8 > 2$, or adding $-8 < -2$, the result is $4 = 4$.

Subtracting $8 > 1$, or adding $-8 < -1$, the result is $4 < 5$.

456. Principle 6.—*If each member of an inequality is subtracted from the corresponding member of an equation, the resulting inequality will subsist in the contrary sense.*

For, let $a > b$ and let c be any number.

Then, § 446, $a - b$ is a positive number.

Since a number is diminished by subtracting a positive number from it,
$$c - (a - b) < c.$$

Transposing,
$$c - a < c - b.$$

That is, if each member of the inequality $a > b$ is subtracted from the corresponding member of the equation $c = c$, the result is an inequality subsisting in a contrary sense.

457. Principle 7.—*If $a > b$ and $b > c$, then $a > c$.*

For, § 446, $a - b$ is positive and $b - c$ is positive.

Therefore, $(a - b) + (b - c)$ is positive;

that is, simplifying, $a - c$ is positive.

Hence, § 446, $a > c$.

Note.—In a similar manner, it may be shown that if $a < b$ and $b < c$, then $a < c$.

458. Principle 8.—*If the corresponding members of two inequalities subsisting in the same sense are multiplied together, the result will be an inequality subsisting in the same sense, provided all the members are positive.*

For, let $a > b$ and $c > d$, a, b, c, and d being positive.

Multiplying the first inequality by c and the second by b, Prin. 2,
$$ac > bc \text{ and } bc > bd.$$

Hence, Prin. 7,
$$ac > bd.$$

otes. — 1. When some of the members are negative, the result may n inequality subsisting in the same or in a contrary sense, or it may 1 equation.

ius, take the inequality $12 > 6$.

ultiplying by $-2 > -5$, $-2 > -3$, and $-2 > -4$, the results are, ctively, $-24 > -30$, $-24 < -18$, and $-24 = -24$.

The *quotient* of two inequalities, member by member, may have its nember greater than, equal to, or less than, its second.

us, take the inequality $12 > 6$.

viding by $3 > 2$, $4 > 2$, and $-2 > -3$, the results are, respectively, $3 = 3$, and $-6 < -2$.

EXERCISES

). 1. Find the values of x in the inequality $3x - 10 > 11$.

SOLUTION

$$3x - 10 > 11.$$

n. 1 or 3, $\quad 3x > 21.$

n. 2, $\quad x > 7.$

erefore, for all values of x greater than 7, the inequality is true; that *inferior limit* of x is 7.

Find the values of x in the simultaneous inequalities 5 < 38 and $4x < 7x - 18$.

SOLUTION

$$3x + 5 < 38. \qquad (1)$$

$$4x < 7x - 18. \qquad (2)$$

nsposing in (1), Prin. 3, $\quad 3x < 33.$

'rin. 2, $\quad x < 11.$

nsposing in (2), Prin. 3, $\quad -3x < -18.$

'rin. 2, $\quad x > 6.$

result shows that the given inequalities are satisfied simultaneously value of x *between* 6 and 11; that is, the *inferior limit* of x is 6, *e superior limit* 11.

Find the limits of x in each of the following:

3. $6x-5>13.$

4. $5x-1<6x+4.$

5. $3x-\frac{1}{2}x<30.$

6. $4x+1<6x-11.$

7. $\begin{cases} 4x-11>\frac{1}{3}x, \\ 20-2x>10. \end{cases}$

8. $\begin{cases} 3-4x<7, \\ 5x+10<20. \end{cases}$

9. $x+\dfrac{2x}{3}+\dfrac{5x}{6}>25$ and $<30.$

10. Find the limits of x and y in $3x-y>-14$ and $x+2y=0.$

Solution

$$\begin{cases} 3x-y>-14, & (1) \\ x+2y=0. & (2) \end{cases}$$

Multiplying (1) by 2, $\quad 6x-2y>-28. \quad (3)$

Adding (2) and (3), $\quad 7x>-28.$

Dividing by 7, $\quad x>-4. \quad (4)$

Multiplying (2) by 3, $\quad 3x+6y=0. \quad (5)$

Subtracting (5) from (1), $\quad -7y>-14.$

Dividing by -7, $\quad y<2.$

That is, x is greater than -4, and y is less than 2.

Find the limits of x and y in the following, and, if possible, one positive integral value for each unknown number:

11. $\begin{cases} 2x-3y<2, \\ 2x+5y=25. \end{cases}$

12. $\begin{cases} 3x+2y=42, \\ 3x-\dfrac{y}{7}>16. \end{cases}$

13. $\begin{cases} x+y=10, \\ 4x<3y. \end{cases}$

14. $\begin{cases} y=3x+4, \\ 25<2y+3x. \end{cases}$

15. $\begin{cases} y-x>9, \\ \dfrac{7x}{20}+\dfrac{y}{15}=9. \end{cases}$

16. $\begin{cases} x>y+4, \\ x-2y=8. \end{cases}$

17. Find the limits of x in $x^2 + 3x > 28$.

Solution

$$x^2 + 3x > 28.$$

ranspoing, Prin. 3, $\quad x^2 + 3x - 28 > 0.$

actoring, $\quad (x-4)(x+7) > 0.$

hat is, $(x-4)(x+7)$ is positive.

ince $(x-4)(x+7)$ is positive, either both factors are positive or are negative. Both factors are positive when $x > 4$; both factors negative when $x < -7$.

ence, x can have all values except 4 and -7 and intermediate values.

ind the limits of x in each of the following:

3. $x^2 + 3x > 10$. 22. $x^2 > 9x - 18$.

). $x^2 + 8x > 20$. 23. $x^2 + 40x > 3(4x - 25)$.

). $x^2 + 5x > 24$. 24. $x^2 + bx > ax + ab$.

l. $(x-2)(3-x) > 0$. 25. $(x-3)(5-x) > 0$.

3. If a and b are positive and unequal, prove that

$$a^2 + b^2 > 2ab.$$

Proof

Vhether $a - b$ is positive or negative, $(a-b)^2$ is positive; and since d b are unequal, $\quad (a-b)^2 > 0$;

is, $\quad a^2 - 2ab + b^2 > 0.$

ransposing, Prin. 3, $\quad a^2 + b^2 > 2ab.$

ote. — If $a = b$, it is evident that $a^2 + b^2 = 2ab$.

7. When a and b are positive, which is the greater,

$$\frac{a+b}{a+2b} \text{ or } \frac{a+2b}{a+3b}?$$

If a, b, and c are positive and unequal:

28. Prove that $a^2 + b^2 + c^2 > ab + ac + bc$.

29. Prove that $a^3 + b^3 > a^2b + ab^2$.

30. Which is the greater, $\frac{a^2 + b^2}{a + b}$ or $\frac{a^3 + b^3}{a^2 + b^2}$?

31. Prove that $\frac{a}{3b} + \frac{3b}{4a} > 1$, except when $2a = 3b$.

32. Prove that $(a - 2b)(4b - a) < b^2$, except when $a = 3b$.

33. Prove that $a^3 + b^3 + c^3 > 3abc$.

34. Prove that the sum of any positive real number (except 1) and its reciprocal is always greater than 2.

35. Prove that a positive proper fraction is increased by adding the same positive number to each of its terms.

36. Find the smallest whole number such that $\frac{1}{2}$ of it decreased by 1 is greater than $\frac{1}{3}$ of it increased by 3.

37. If 5 times the number of pupils in a certain department, plus 25, is less than 6 times the number, minus 74; and if twice the number, plus 50, is greater than 3 times the number, minus 51, how many pupils are there in the department?

38. At least how many dollars must A and B each have that 5 times A's money, plus B's money, shall be more than \$51, and 3 times A's money, minus B's money, shall be \$21?

39. Four times the number of passenger trains entering a certain city daily, minus 136, is less than three times the number, plus 24; and 4 times the number, plus 63, is less than 5 times the number, minus 95. How many passenger trains enter the city each day?

40. Three times the number of soldiers in a full regiment, less 593, is less than 2 times the number, plus 608; and 8 times the number, minus 577, is less than 9 times the number, minus 1776. How many soldiers are there in a full regiment?

RATIO AND PROPORTION

RATIO

[3]60. The relation of two numbers that is expressed by the [quo]tient of the first divided by the second is called their **ratio**.

[3]61. The **sign of ratio** is a colon (:).

[A] ratio is expressed also in the form of a fraction.

[T]he ratio of a to b is written $a : b$ or $\frac{a}{b}$.

[T]he colon is sometimes regarded as derived from the sign of division [o]mitting the line.

[3]62. To compare two quantities they must be expressed in [ter]*ms of a common unit.*

[T]hus, to indicate the ratio of 20¢ to \$1, both quantities must be ex[pres]sed either in cents or in dollars, as 20¢ : 100¢ or \$$\frac{1}{5}$: \$1.
[Th]ere can be no ratio between 2 lb. and 3 ft.

[T]he ratio of two quantities is the *ratio* of their *numerical* [mea]*sures.*

[T]hus, the ratio of 4 rods to 5 rods is the ratio of 4 to 5.

[3]63. The first term of a ratio is called the **antecedent**, and [the] second, the **consequent**. Both terms form a **couplet**.
[T]he antecedent corresponds to a dividend or numerator; the [con]sequent, to a divisor or denominator.

[I]n the ratio $a : b$, or $\frac{a}{b}$, a is the antecedent, b the consequent, and [the] terms a and b form a couplet.

464. A ratio is said to be a ratio of **greater inequality,** a ratio of **equality,** or a ratio of **less inequality,** according as the antecedent is *greater than, equal to,* or *less than* the consequent.

Thus, when a and b are positive numbers, $a : b$ is a ratio of greater inequality, if $a > b$; a ratio of equality, if $a = b$; and a ratio of less inequality, if $a < b$.

465. The ratio of the reciprocals of two numbers is called the **reciprocal,** or **inverse, ratio** of the numbers.

It may be expressed by interchanging the terms of the couplet.

The inverse ratio of a to b is $\frac{1}{a} : \frac{1}{b}$. Since $\frac{1}{a} \div \frac{1}{b} = \frac{b}{a}$, the inverse ratio of a to b may be written $\frac{b}{a}$, or $b : a$.

466. The ratio of the squares of two numbers is called their **duplicate** ratio; the ratio of their cubes, their **triplicate** ratio.

The duplicate ratio of a to b is $a^2 : b^2$; the triplicate ratio, $a^3 : b^3$.

467. If the ratio of two numbers can be expressed by the ratio of two integers, the numbers are called **commensurable numbers,** and their ratio a **commensurable ratio.**

468. If the ratio of two numbers cannot be expressed by the ratio of two integers, the numbers are called **incommensurable numbers,** and their ratio an **incommensurable ratio.**

The ratio $\sqrt{2} : 3 = \frac{\sqrt{2}}{3} = \frac{1.414213+}{3}$ cannot be expressed by any two integers, because there is no number that, used as a common *measure,* will be contained in both $\sqrt{2}$ and 3 an integral number of times. Hence, $\sqrt{2}$ and 3 are incommensurable, and $\sqrt{2} : 3$ is an incommensurable ratio.

It is evident that by continuing the process of extracting the square root of 2, the ratio $\sqrt{2} : 3$ may be expressed by two integers to any desired degree of approximation, but never with absolute accuracy.

Properties of Ratios

469. It is evident from the definition of a ratio that *ratios have the same properties as fractions;* that is, they may be *reduced to higher or lower terms, added, subtracted, etc.* Hence,

PRINCIPLES. — 1. *Multiplying or dividing both terms of a ratio the same number does not change the value of the ratio.*

2. *Multiplying the antecedent or dividing the consequent of a io by any number multiplies the ratio by that number.*

3. *Dividing the antecedent or multiplying the consequent by number divides the ratio by that number.*

470. If the same positive number is added to both terms of raction, the value of the fraction will be nearer 1 than before, ether the fraction is improper or proper. The correspond- principle for ratios follows:

PRINCIPLE 4. — *A ratio of greater inequality is decreased and atio of less inequality is increased by adding the same positive mber to each of its terms.*

For, given the positive numbers a, b, and c, and the ratio $\frac{a}{b}$.

1. *When $a > b$, it is to be proved that* $\frac{a+c}{b+c} < \frac{a}{b}$.

$$\frac{a+c}{b+c} - \frac{a}{b} = \frac{c(b-a)}{b(b+c)}.$$

Since $b - a$ is negative, because $a > b$,

$$\frac{c(b-a)}{b(b+c)} \text{ is negative;}$$

erefore, $\qquad \frac{a+c}{b+c} - \frac{a}{b}$ is negative.

Hence, § 446, $\qquad \frac{a+c}{b+c} < \frac{a}{b}$.

2. *When $a < b$, it is to be proved that* $\frac{a+c}{b+c} > \frac{a}{b}$.

Proceeding by the method used in 1, since $a < b$ it may be shown

it $\qquad \frac{a+c}{b+c} - \frac{a}{b}$ is positive.

Hence, § 446, $\qquad \frac{a+c}{b+c} > \frac{a}{b}$.

EXERCISES

471. 1. What is the ratio of $8m$ to $4m$? of $4m$ to $8m$?

2. Express the ratio $6:9$ in its lowest terms; the ratio $12x:16y$; $am:bm$; $20ab:10bc$; $(m+n):(m^2-n^2)$.

3. Which is the greater ratio, $2:3$ or $3:4$? $4:9$ or $2:5$?

4. What is the ratio of $\frac{1}{2}$ to $\frac{1}{4}$? $\frac{1}{2}$ to $\frac{1}{3}$? $\frac{2}{3}$ to $\frac{3}{4}$?

SUGGESTION. — When fractions have a common denominator, they have the ratio of their numerators.

5. What is the inverse ratio of $3:10$? of $12:7$?

6. Write the duplicate ratio of $2:3$; of $4:5$; the triplicate ratio of $1:2$; of $3:4$.

Reduce to lowest terms the ratios expressed by:

7. $10:2$.	10. $3:27$.	13. $\frac{12}{24}$.	16. $75 \div 100$.
8. $12:6$.	11. $4:40$.	14. $\frac{16}{48}$.	17. $60 \div 120$.
9. $16:4$.	12. $9:72$.	15. $\frac{60}{80}$.	18. $80 \div 240$.

19. What is the ratio of 15 days to 30 days? of 21 days to 1 week? of 1 rod to 1 mile?

Find the value of each of the following ratios:

20. $\frac{4}{5}x : \frac{2}{5}x^2$.	23. $2\frac{1}{2} : 7\frac{1}{2}$.	26. $a^2b^3x^4 : a^4b^2x^2$.
21. $\frac{3}{4}ab : \frac{1}{2}ac$.	24. $.7\,m : .8\,n$.	27. $(x^2-y^2):(x-y)^2$.
22. $\frac{5}{8}x^2y^2 : \frac{1}{4}xy$.	25. $.4\,x^2 : 10\,x^3$.	28. $(a^3-1):(a^2+a+1)$.

29. Two numbers are in the ratio of $4:5$. If 9 is subtracted from each, find the ratio of the remainders.

30. Change each to a ratio whose antecedent shall be 1:

$$5:20;\quad 3x:12x;\quad \tfrac{3}{2}:\tfrac{5}{6};\quad .4:1.2.$$

31. Reduce the ratios $a:b$ and $x:y$ to ratios having the same consequent.

32. When the antecedent is $6x$ and the ratio is $\frac{1}{3}$, what is the consequent?

33. Given the ratio $\frac{2}{3}$ and a positive number x. Prove that $\frac{2+x}{3+x} > \frac{2}{3}$ by subtracting one ratio from the other.

SUGGESTION. — Proceed as in § 470.

34. The capital stock of a street railway company was \$7,500,000, the gross earnings for a year \$1,500,000, and the net earnings \$600,000. Find the ratio of gross earnings to capital stock; of net earnings to gross earnings; of net earnings to capital stock.

PROPORTION

472. An equality of ratios is called a **proportion**.

$3:10 = 6:20$ and $a:x = b:y$ are proportions.

The double colon (: :) is often used instead of the sign of equality.

The double colon has been supposed to represent the extremities of the lines that form the sign of equality.

The proportion $a:b=c:d$, or $a:b::c:d$, is read, 'the ratio of a to b is equal to the ratio of c to d,' or 'a is to b as c is to d.'

473. In a proportion, the first and fourth terms are called the **extremes**, and the second and third terms, the **means**.

In $a:b=c:d$, a and d are the extremes, b and c are the means.

474. Since a proportion is an equality of ratios each of which may be expressed as a fraction, a proportion may be expressed as an equation each member of which is a fraction. Hence, it follows that:

GENERAL PRINCIPLE. — *The changes that may be made in a proportion without destroying the equality of its ratios correspond to the changes that may be made in the members of an equation without destroying their equality and in the terms of a fraction without altering the value of the fraction.*

Properties of Proportions

475. PRINCIPLE 1. — *In any proportion, the product of the extremes is equal to the product of the means.*

For, given $a:b = c:d,$

or $\frac{a}{b} = \frac{c}{d}.$

Clearing of fractions, $ad = bc.$

Test the following by principle 1 to find whether they are true proportions:

1. $6:16 = 3:8.$ 2. $\frac{12}{15} = \frac{16}{20}.$ 3. $7:8 = 10:12.$

476. In the proportion $a:m = m:b$, m is called a **mean proportional** between a and b.

By Prin. 1, $m^2 = ab;$

$$\therefore m = \sqrt{ab}.$$

Hence, *a mean proportional between two numbers is equal to the square root of their product.*

1. Show that the mean proportional between 3 and 12 is either 6 or -6. Write both proportions.

2. Find two mean proportionals between 4 and 25.

477. PRINCIPLE 2. — *Either extreme of a proportion is equal to the product of the means divided by the other extreme.*

Either mean is equal to the product of the extremes divided by the other mean.

For, given $a:b = c:d.$

By Prin. 1, $ad = bc.$

Solving for a, d, b, and c, in succession, Ax. 4,

$$a = \frac{bc}{d},\ d = \frac{bc}{a},\ b = \frac{ad}{c},\ c = \frac{ad}{b}.$$

1. *Solve* the proportion $3:4 = x:20$, for x.

2. *Solve* the proportion $x:a = 2m:n$, for x.

3. If $a : b = b : c$, the term c is called a **third proportional** to a nd b. Find a third proportional to 6 and 2.

4. In the proportion $a : b = c : d$, the term d is called **a fourth roportional** to a, b, and c. Find a fourth proportional to $\frac{1}{2}$, $\frac{1}{3}$, nd $\frac{1}{4}$.

478. Principle 3. — *If the product of two numbers is equal the product of two other numbers, one pair of them may be ade the extremes and the other pair the means of a proportion.*

For, given $\qquad ad = bc.$

Dividing by bd, Ax. 4, $\qquad \frac{a}{b} = \frac{c}{d}$;

at is, $\qquad a : b = c : d.$

By dividing both members of the given equation, or of $bc = ad$, by e proper numbers, various proportions may be obtained; but in all them a and d will be the extremes and b and c the means, or *vice rsa*, as illustrated in the proofs of principles 4 and 5.

1. If a men can do a piece of work in x days, and if b men n do the same work in y days, the number of days' work for e man may be expressed by either ax or by. Form a pro- rtion between a, b, x, and y.

2. The formula $\qquad pd = WD \qquad$ (See p. 173)

presses the physical law that, when a lever just balances, e product of the numerical measures of the power and its stance from the fulcrum is equal to the product of the merical measures of the weight and its distance from the lcrum. Express this law by means of a proportion.

479. Principle 4. — *If four numbers are in proportion, the tio of the antecedents is equal to the ratio of the consequents;* at is, *the numbers are in proportion by* **alternation.**

For, given $\qquad a : b = c : d.$

Then, Prin. 1, $\qquad ad = bc.$

Dividing by cd, Ax. 4, $\qquad \frac{a}{c} = \frac{b}{d}$;

at is, $\qquad a : c = b : d.$

Find the limits of x in each of the following:

3. $6x-5>13.$

4. $5x-1<6x+4.$

5. $3x-\frac{1}{2}x<30.$

6. $4x+1<6x-11.$

7. $\begin{cases}4x-11>\frac{1}{3}x,\\ 20-2x>10.\end{cases}$

8. $\begin{cases}3-4x<7,\\ 5x+10<20.\end{cases}$

9. $x+\frac{2x}{3}+\frac{5x}{6}>25$ and $<30.$

10. Find the limits of x and y in $3x-y>-14$ and $x+2y=0.$

SOLUTION

$$\begin{cases}3x-y>-14, & (1)\\ x+2y=0. & (2)\end{cases}$$

Multiplying (1) by 2, $6x-2y>-28.$ (3)

Adding (2) and (3), $7x>-28.$

Dividing by 7, $x>-4.$ (4)

Multiplying (2) by 3, $3x+6y=0.$ (5)

Subtracting (5) from (1), $-7y>-14.$

Dividing by -7, $y<2.$

That is, x is greater than -4, and y is less than 2.

Find the limits of x and y in the following, and, if possible, one positive integral value for each unknown number:

11. $\begin{cases}2x-3y<2,\\ 2x+5y=25.\end{cases}$

12. $\begin{cases}3x+2y=42,\\ 3x-\frac{y}{7}>16.\end{cases}$

13. $\begin{cases}x+y=10,\\ 4x<3y.\end{cases}$

14. $\begin{cases}y=3x+4,\\ 25<2y+3x.\end{cases}$

15. $\begin{cases}y-x>9,\\ \frac{7x}{20}+\frac{y}{15}=9.\end{cases}$

16. $\begin{cases}x>y+4,\\ x-2y=8.\end{cases}$

7. Find the limits of x in $x^2 + 3x > 28$.

Solution

$$x^2 + 3x > 28.$$

ransposing, Prin. 3, $x^2 + 3x - 28 > 0$.

ıctoring, $(x-4)(x+7) > 0$.

ıat is, $(x - 4)(x + 7)$ is positive.

nce $(x-4)(x+7)$ is positive, either both factors are positive or are negative. Both factors are positive when $x > 4$; both factors ıegative when $x < -7$.

ence, x can have all values except 4 and -7 and intermediate values.

ind the limits of x in each of the following:

3. $x^2 + 3x > 10$.

9. $x^2 + 8x > 20$.

0. $x^2 + 5x > 24$.

1. $(x-2)(3-x) > 0$.

22. $x^2 > 9x - 18$.

23. $x^2 + 40x > 3(4x - 25)$.

24. $x^2 + bx > ax + ab$.

25. $(x-3)(5-x) > 0$.

3. If a and b are positive and unequal, prove that

$$a^2 + b^2 > 2ab.$$

Proof

Whether $a - b$ is positive or negative, $(a-b)^2$ is positive; and since d b are unequal, $(a-b)^2 > 0$;

is, $a^2 - 2ab + b^2 > 0$.

ransposing, Prin. 3, $a^2 + b^2 > 2ab$.

OTE. — If $a = b$, it is evident that $a^2 + b^2 = 2ab$.

7. When a and b are positive, which is the greater,

$$\frac{a+b}{a+2b} \text{ or } \frac{a+2b}{a+3b}?$$

If a, b, and c are positive and unequal:

28. Prove that $a^2 + b^2 + c^2 > ab + ac + bc$.

29. Prove that $a^3 + b^3 > a^2b + ab^2$.

30. Which is the greater, $\frac{a^2 + b^2}{a + b}$ or $\frac{a^3 + b^3}{a^2 + b^2}$?

31. Prove that $\frac{a}{3b} + \frac{3b}{4a} > 1$, except when $2a = 3b$.

32. Prove that $(a - 2b)(4b - a) < b^2$, except when $a = 3b$.

33. Prove that $a^3 + b^3 + c^3 > 3abc$.

34. Prove that the sum of any positive real number (except 1) and its reciprocal is always greater than 2.

35. Prove that a positive proper fraction is increased by adding the same positive number to each of its terms.

36. Find the smallest whole number such that $\frac{1}{2}$ of it decreased by 1 is greater than $\frac{1}{3}$ of it increased by 3.

37. If 5 times the number of pupils in a certain department, plus 25, is less than 6 times the number, minus 74; and if twice the number, plus 50, is greater than 3 times the number, minus 51, how many pupils are there in the department?

38. At least how many dollars must A and B each have that 5 times A's money, plus B's money, shall be more than \$51, and 3 times A's money, minus B's money, shall be \$21?

39. Four times the number of passenger trains entering a certain city daily, minus 136, is less than three times the number, plus 24; and 4 times the number, plus 63, is less than 5 times the number, minus 95. How many passenger trains enter the city each day?

40. Three times the number of soldiers in a full regiment, less 593, is less than 2 times the number, plus 608; and 8 times the number, minus 577, is less than 9 times the number, minus 1776. How many soldiers are there in a full regiment?

RATIO AND PROPORTION

RATIO

). The relation of two numbers that is expressed by the ent of the first divided by the second is called their **ratio.**

.. The **sign of ratio** is a colon (:).

:atio is expressed also in the form of a fraction.

: ratio of a to b is written $a : b$ or $\frac{a}{b}$.

: colon is sometimes regarded as derived from the sign of division itting the line.

!. To compare two quantities they must be expressed in *of a common unit.*

ıs, to indicate the ratio of 20 ¢ to \$ 1, both quantities must be ex-d either in cents or in dollars, as 20 ¢ : 100 ¢ or \$ $\frac{1}{5}$: \$ 1.
re can be no ratio between 2 lb. and 3 ft.

e ratio of two quantities is the *ratio* of their *numerical* ıres.

ıs, the ratio of 4 rods to 5 rods is the ratio of 4 to 5.

!. The first term of a ratio is called the **antecedent,** and ɜcond, the **consequent.** Both terms form a **couplet.**

e antecedent corresponds to a dividend or numerator; the quent, to a divisor or denominator.

the ratio $a : b$, or $\frac{a}{b}$, a is the antecedent, b the consequent, and *ms a and b form* a couplet.

464. A ratio is said to be a ratio of **greater inequality,** a ratio of **equality,** or a ratio of **less inequality,** according as the antecedent is *greater than, equal to,* or *less than* the consequent.

Thus, when a and b are positive numbers, $a : b$ is a ratio of greater inequality, if $a > b$; a ratio of equality, if $a = b$; and a ratio of less inequality, if $a < b$.

465. The ratio of the reciprocals of two numbers is called the **reciprocal,** or **inverse, ratio** of the numbers.

It may be expressed by interchanging the terms of the couplet.

The inverse ratio of a to b is $\frac{1}{a} : \frac{1}{b}$. Since $\frac{1}{a} \div \frac{1}{b} = \frac{b}{a}$, the inverse ratio of a to b may be written $\frac{b}{a}$, or $b : a$.

466. The ratio of the squares of two numbers is called their **duplicate** ratio; the ratio of their cubes, their **triplicate** ratio.

The duplicate ratio of a to b is $a^2 : b^2$; the triplicate ratio, $a^3 : b^3$.

467. If the ratio of two numbers can be expressed by the ratio of two integers, the numbers are called **commensurable numbers,** and their ratio a **commensurable ratio.**

468. If the ratio of two numbers cannot be expressed by the ratio of two integers, the numbers are called **incommensurable numbers,** and their ratio an **incommensurable ratio.**

The ratio $\sqrt{2} : 3 = \frac{\sqrt{2}}{3} = \frac{1.414213+}{3}$ cannot be expressed by any two integers, because there is no number that, used as a common *measure,* will be contained in both $\sqrt{2}$ and 3 an integral number of times. Hence, $\sqrt{2}$ and 3 are incommensurable, and $\sqrt{2} : 3$ is an incommensurable ratio.

It is evident that by continuing the process of extracting the square root of 2, the ratio $\sqrt{2} : 3$ may be expressed by two integers to any desired degree of approximation, but never with absolute accuracy.

Properties of Ratios

469. It is evident from the definition of a ratio that ratios *have the* same properties as fractions; that is, they may be *reduced to higher or lower* terms, added, subtracted, etc. Hence,

Principles. — 1. *Multiplying or dividing both terms of a ratio by the same number does not change the value of the ratio.*

2. *Multiplying the antecedent or dividing the consequent of a ratio by any number multiplies the ratio by that number.*

3. *Dividing the antecedent or multiplying the consequent by any number divides the ratio by that number.*

470. If the same positive number is added to both terms of a fraction, the value of the fraction will be nearer 1 than before, whether the fraction is improper or proper. The corresponding principle for ratios follows:

Principle 4. — *A ratio of greater inequality is decreased and a ratio of less inequality is increased by adding the same positive number to each of its terms.*

For, given the positive numbers a, b, and c, and the ratio $\frac{a}{b}$.

1. *When $a > b$, it is to be proved that* $\frac{a+c}{b+c} < \frac{a}{b}$.

$$\frac{a+c}{b+c} - \frac{a}{b} = \frac{c(b-a)}{b(b+c)}.$$

Since $b - a$ is negative, because $a > b$,

$$\frac{c(b-a)}{b(b+c)} \text{ is negative;}$$

Therefore, $\qquad \frac{a+c}{b+c} - \frac{a}{b}$ is negative.

Hence, § 446, $\qquad \frac{a+c}{b+c} < \frac{a}{b}$.

2. *When $a < b$, it is to be proved that* $\frac{a+c}{b+c} > \frac{a}{b}$.

Proceeding by the method used in 1, since $a < b$ it may be shown

that $\qquad \frac{a+c}{b+c} - \frac{a}{b}$ is positive.

Hence, § 446, $\qquad \frac{a+c}{b+c} > \frac{a}{b}$.

EXERCISES

471. 1. What is the ratio of $8\,m$ to $4\,m$? of $4\,m$ to $8\,m$?

2. Express the ratio $6:9$ in its lowest terms; the ratio $12\,x:16\,y$; $am:bm$; $20\,ab:10\,bc$; $(m+n):(m^2-n^2)$.

3. Which is the greater ratio, $2:3$ or $3:4$? $4:9$ or $2:5$?

4. What is the ratio of $\frac{1}{2}$ to $\frac{1}{4}$? $\frac{1}{2}$ to $\frac{1}{3}$? $\frac{2}{3}$ to $\frac{3}{4}$?

SUGGESTION. — When fractions have a common denominator, they have the ratio of their numerators.

5. What is the inverse ratio of $3:10$? of $12:7$?

6. Write the duplicate ratio of $2:3$; of $4:5$; the triplicate ratio of $1:2$; of $3:4$.

Reduce to lowest terms the ratios expressed by:

7. $10:2$.	10. $3:27$.	13. $\frac{12}{24}$.	16. $75 \div 100$.
8. $12:6$.	11. $4:40$.	14. $\frac{16}{48}$.	17. $60 \div 120$.
9. $16:4$.	12. $9:72$.	15. $\frac{60}{80}$.	18. $80 \div 240$.

19. What is the ratio of 15 days to 30 days? of 21 days to 1 week? of 1 rod to 1 mile?

Find the value of each of the following ratios:

20. $\frac{4}{5}\,x:\frac{2}{5}\,x^2$.	23. $2\frac{1}{2}:7\frac{1}{2}$.	26. $a^2b^3x^4:a^4b^2x^2$.
21. $\frac{3}{4}\,ab:\frac{1}{2}\,ac$.	24. $.7\,m:.8\,n$.	27. $(x^2-y^2):(x-y)^2$.
22. $\frac{5}{8}\,x^2y^2:\frac{1}{4}\,xy$.	25. $.4\,x^2:10\,x^3$.	28. $(a^3-1):(a^2+a+1)$.

29. Two numbers are in the ratio of $4:5$. If 9 is subtracted from each, find the ratio of the remainders.

30. Change each to a ratio whose antecedent shall be 1:

$$5:20;\quad 3\,x:12\,x;\quad \tfrac{3}{2}:\tfrac{5}{6};\quad .4:1.2.$$

31. Reduce the ratios $a:b$ and $x:y$ to ratios having the same consequent.

32. When the antecedent is $6\,x$ and the ratio is $\frac{1}{8}$, what is the consequent?

33. Given the ratio $\frac{2}{3}$ and a positive number x. Prove that $\frac{+x}{+x} > \frac{2}{3}$ by subtracting one ratio from the other.

SUGGESTION. — Proceed as in § 470.

34. The capital stock of a street railway company was 7,500,000, the gross earnings for a year $1,500,000, and the et earnings $600,000. Find the ratio of gross earnings to ıpital stock; of net earnings to gross earnings; of net earnıgs to capital stock.

PROPORTION

472. An equality of ratios is called a **proportion**.

$3:10 = 6:20$ and $a:x = b:y$ are proportions.

The double colon (: :) is often used instead of the sign of quality.

The double colon has been supposed to represent the extremities of ıe lines that form the sign of equality.

The proportion $a:b=c:d$, or $a:b::c:d$, is read, 'the ratio f a to b is equal to the ratio of c to d,' or 'a is to b as is to d.'

473. In a proportion, the first and fourth terms are called e **extremes**, and the second and third terms, the **means**.

In $a:b = c:d$, a and d are the extremes, b and c are the means.

474. Since a proportion is an equality of ratios each of hich may be expressed as a fraction, a proportion may be pressed as an equation each member of which is a fraction. Hence, it follows that:

GENERAL PRINCIPLE. — *The changes that may be made in a roportion without destroying the equality of its ratios correspond the changes that may be made in the members of an equation ithout destroying their equality and in the terms of a fraction ithout altering the value of the fraction.*

Properties of Proportions

475. Principle 1. — *In any proportion, the product of the extremes is equal to the product of the means.*

For, given $a:b = c:d,$

or $\frac{a}{b} = \frac{c}{d}.$

Clearing of fractions, $ad = bc.$

Test the following by principle 1 to find whether they are true proportions:

1. $6:16 = 3:8.$ 2. $\frac{12}{15} = \frac{16}{20}.$ 3. $7:8 = 10:12.$

476. In the proportion $a:m = m:b$, m is called a **mean proportional** between a and b.

By Prin. 1, $m^2 = ab;$

$\therefore\ m = \sqrt{ab}.$

Hence, *a mean proportional between two numbers is equal to the square root of their product.*

1. Show that the mean proportional between 3 and 12 is either 6 or -6. Write both proportions.

2. Find two mean proportionals between 4 and 25.

477. Principle 2. — *Either extreme of a proportion is equal to the product of the means divided by the other extreme.*

Either mean is equal to the product of the extremes divided by the other mean.

For, given $a:b = c:d.$

By Prin. 1, $ad = bc.$

Solving for a, d, b, and c, in succession, Ax. 4,

$$a = \frac{bc}{d},\ d = \frac{bc}{a},\ b = \frac{ad}{c},\ c = \frac{ad}{b}.$$

1. Solve the proportion $3:4 = x:20$, for x.

2. Solve the proportion $x:a = 2m:n$, for x.

3. If $a : b = b : c$, the term c is called a **third proportional** to a ıd b. Find a third proportional to 6 and 2.

4. In the proportion $a : b = c : d$, the term d is called **a fourth** **:oportional** to a, b, and c. Find a fourth proportional to $\frac{1}{2}$, $\frac{1}{8}$, ıd $\frac{1}{4}$.

478. PRINCIPLE 3. — *If the product of two numbers is equal the product of two other numbers, one pair of them may be ade the extremes and the other pair the means of a proportion.*

For, given $\qquad ad = bc.$

Dividing by bd, Ax. 4, $\qquad \frac{a}{b} = \frac{c}{d}$;

ıat is, $\qquad a : b = c : d.$

By dividing both members of the given equation, or of $bc = ad$, by ıe proper numbers, various proportions may be obtained; but in all : them a and d will be the extremes and b and c the means, or *vice* *:rsa*, as illustrated in the proofs of principles 4 and 5.

1. If a men can do a piece of work in x days, and if b men ın do the same work in y days, the number of days' work for ne man may be expressed by either ax or by. Form a proortion between a, b, x, and y.

2. The formula $\qquad pd = WD \qquad$ (See p. 173)

:presses the physical law that, when a lever just balances, e product of the numerical measures of the power and its stance from the fulcrum is equal to the product of the merical measures of the weight and its distance from the lcrum. Express this law by means of a proportion.

479. PRINCIPLE 4. — *If four numbers are in proportion, the io of the antecedents is equal to the ratio of the consequents;* ıt is, *the numbers are in proportion by* **alternation.**

For, given $\qquad a : b = c : d.$

Then, Prin. 1, $\qquad ad = bc.$

Dividing by cd, Ax. 4, $\qquad \frac{a}{c} = \frac{b}{d}$;

t is, $\qquad a : c = b : d.$

480. PRINCIPLE 5. — *If four numbers are in proportion, the ratio of the second to the first is equal to the ratio of the fourth to the third;* that is, *the numbers are in proportion by* **inversion.**

For, given $a : b = c : d.$

Then, Prin. 1, $ad = bc.$

$\therefore bc = ad.$

Dividing by ac, Ax. 4, $\frac{b}{a} = \frac{d}{c};$

that is, $b : a = d : c.$

481. PRINCIPLE 6. — *If four numbers are in proportion, the sum of the terms of the first ratio is to either term of the first ratio as the sum of the terms of the second ratio is to the corresponding term of the second ratio;* that is, *the numbers are in proportion by* **composition.**

For, given $a : b = c : d,$

or $\frac{a}{b} = \frac{c}{d}.$

Then, $\frac{a}{b} + 1 = \frac{c}{d} + 1,$

or $\frac{a+b}{b} = \frac{c+d}{d};$

that is, $a + b : b = c + d : d.$

Similarly, taking the given proportion by inversion (Prin. 5), and adding 1 to both members, we obtain

$$a + b : a = c + d : c.$$

482. PRINCIPLE 7. — *If four numbers are in proportion, the difference between the terms of the first ratio is to either term of the first ratio as the difference between the terms of the second ratio is to the corresponding term of the second ratio;* that is, *the numbers are in proportion by* **division.**

For, in the proof of Prin. 6, if 1 is *subtracted* instead of added, the following proportions are obtained:

$$a - b : b = c - d : d,$$

and $$a - b : a = c - d : c.$$

483. Principle 8.—*If four numbers are in proportion, the um of the terms of the first ratio is to their difference as the sum f the terms of the second ratio is to their difference;* that is, *the umbers are in proportion by* **composition and division.**

For, given $$a : b = c : d.$$

By Prin. 6, $$\frac{a+b}{b} = \frac{c+d}{d}. \qquad (1)$$

By Prin 7, $$\frac{a-b}{b} = \frac{c-d}{d}. \qquad (2)$$

Dividing (1) by (2), Ax. 4, $$\frac{a+b}{a-b} = \frac{c+d}{c-d};$$

at is, $$a+b : a-b = c+d : c-d.$$

484. Principle 9.—*If four numbers are in proportion, their ke powers, and also their like roots, are in proportion.*

For, given $$a : b = c : d,$$

r $$\frac{a}{b} = \frac{c}{d}.$$

Then, Ax. 6 and § 276, 3, $$\frac{a^n}{b^n} = \frac{c^n}{d^n};$$

at is, $$a^n : b^n = c^n : d^n.$$

Also, Ax. 7 and § 291, regarding only principal roots,

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \frac{\sqrt[n]{c}}{\sqrt[n]{d}};$$

at is, $$\sqrt[n]{a} : \sqrt[n]{b} = \sqrt[n]{c} : \sqrt[n]{d}.$$

485. Principle 10.—*In a proportion, if both terms of a couplet, both antecedents, or both consequents are multiplied or divided the same number, the resulting four numbers form a proportion.*

For, given $$a : b = c : d,$$

$$\frac{a}{b} = \frac{c}{d}.$$

Then, § 195, $$\frac{ma}{mb} = \frac{nc}{nd}, \text{ or } ma : mb = nc : nd.$$

Also, Ax. 3, $$\frac{a}{b} \cdot \frac{m}{n} = \frac{c}{d} \cdot \frac{m}{n}, \text{ or } ma : nb = mc : nd.$$

486. Principle 11. — *The products of corresponding terms of any number of proportions form a proportion.*

For, given $$a : b = c : d,$$
$$k : l = m : n,$$
and $$x : y = z : w.$$

Writing each proportion as a fractional equation, we have

$$\frac{a}{b} = \frac{c}{d}, \ \frac{k}{l} = \frac{m}{n}, \text{ and } \frac{x}{y} = \frac{z}{w}.$$

Multiplying these equations, member by member, Ax. 3, we have

$$\frac{akx}{bly} = \frac{cmz}{dnw};$$

that is, $$akx : bly = cmz : dnw.$$

487. Principle 12. — *If two proportions have a common couplet, the other two couplets will form a proportion.*

For, given $$a : b = x : y,$$
and $$c : d = x : y.$$
Then, Ax. 5, $$a : b = c : d.$$

488. A proportion that consists of three or more equal ratios is called a **multiple proportion.**

$2 : 4 = 3 : 6 = 5 : 10$ and $a : b = c : d = e : f$ are multiple proportions.

489. Principle 13. — *In any multiple proportion the sum of all the antecedents is to the sum of all the consequents as any antecedent is to its consequent.*

For, given $$a : b = c : d = e : f,$$
or $$\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = r, \text{ the value of each ratio.}$$

Then, Ax. 3, $$a = br, \ c = dr, \ e = fr;$$
whence, Ax. 1, $$a + c + e = (b + d + f)r,$$

$$\therefore \frac{a + c + e}{b + d + f} = r = \frac{a}{b} = \frac{c}{d} = \frac{e}{f};$$

that is, $$a + c + e : b + d + f = a : b \text{ or } c : d \text{ or } e : f.$$

. A multiple proportion in which each consequent is ted as the antecedent of the following ratio is called a **ued proportion.**

$=4:8=8:16$ and $a:b=b:c=c:d$ are continued proportions.

. PRINCIPLE 14. — *If three numbers are in continued pro-n, the ratio of the extremes is equal to the square of either ratio.*

, given $\qquad a:b=b:c,$

$$\frac{a}{b}=\frac{b}{c}. \qquad (1)$$

ltiplying by $\frac{a}{b}$, Ax. 3, $\qquad \frac{a^2}{b^2}=\frac{a}{c}, \qquad (2)$

(1) and Prin. 9, $\qquad \frac{a^2}{b^2}=\frac{b^2}{c^2}. \qquad (3)$

(2) and (3), Ax. 5, $\qquad \frac{a}{c}=\frac{a^2}{b^2}=\frac{b^2}{c^2};$

, $\qquad a:c=a^2:b^2=b^2:c^2.$

. PRINCIPLE 15. — *If four numbers are in continued pro-n, the ratio of the extremes is equal to the cube of any of the ratios.*

, given $\qquad a:b=b:c=c:d,$

$$\frac{a}{b}=\frac{b}{c}=\frac{c}{d}. \qquad (1)$$

n, Ax. 5, $\qquad \frac{a}{b}\cdot\frac{b}{c}\cdot\frac{c}{d}=\frac{a}{b}\cdot\frac{a}{b}\cdot\frac{a}{b}=\frac{a^3}{b^3}; \qquad (2)$

, canceling, $\qquad a:d=a^3:b^3. \qquad (3)$

(1) and Prin. 9, $\qquad a^3:b^3=b^3:c^3=c^3:d^3. \qquad (4)$

(4) and (3), Ax. 5. $\qquad a:d=a^3:b^3=b^3:c^3=c^3:d^3.$

EXERCISES

. 1. Find the value of x in the proportion $3:5=x:55$.

UTION. $\qquad 3:5=x:55.$

1. 2, $\qquad x=\frac{3\cdot 55}{5}=33.$

Find the value of x in each of the following proportions:

2. $2:3=6:x$.
3. $5:x=4:3$.
4. $1:x=x:9$.
5. $x+2:x=10:6$.
6. $x:x-1=15:12$.
7. $x+2:x-2=3:1$.
8. Show that a mean proportional between any two numbers having like signs has the sign $\pm$.
9. Find two mean proportionals between $\sqrt{2}$ and $\sqrt{8}$.
10. Find a third proportional to 4 and 6.
11. Find a fourth proportional to 3, 8, and $7\frac{1}{2}$.
12. Find a mean proportional between

$$\frac{x^2-x-6}{x+4} \text{ and } \frac{x^2+x-12}{x+2}.$$

Test to see whether the following are true proportions:

13. $5\frac{1}{2}:3=4:1\frac{1}{2}$.
14. $4:13=2:6\frac{1}{2}$.
15. $5:7x=10:14x$.
16. $2.4a:.8a=6a:2a$.
17. Given $a:b=c:d$,

to prove that $2a+5b:4a-3b=2c+5d:4c-3d$.

Proof. — First form, from the given proportion, a proportion having as antecedents the antecedents of the required proportion.

$$a:b=c:d. \tag{1}$$

Prin. 10, $2a:b=2c:d$.

Prin. 10, $2a:5b=2c:5d$.

Prin. 6, $$2a+5b:2a=2c+5d:2c. \tag{2}$$

Next form from (1) a proportion having as antecedents the consequents of the required proportion.

Prin. 10, $4a:b=4c:d$.

Prin. 10, $4a:3b=4c:3d$.

Prin. 7, $$4a-3b:4a=4c-3d:4c. \tag{3}$$

Prin. 10, $$4a-3b:2a=4c-3d:2c. \tag{4}$$

Next take (2) and (4) by alternation (Prin. 4) and apply Prin. 12 to the results.

Then, $2a+5b:2c+5d=4a-3b:4c-3d$,

or, Prin. 4, $2a+5b:4a-3b=2c+5d:4c-3d$.

When $a:b=c:d$, prove that the following are true proportions:

18. $d:b=c:a$.

19. $c:d=\frac{1}{b}:\frac{1}{a}$.

20. $b^3:d^3=a^3:c^3$.

21. $a^2:b^2c^2=1:d^2$.

22. $ma:\frac{b}{2}=mc:\frac{d}{2}$.

23. $ac:bd=c^2:d^2$.

24. $\sqrt{ad}:\sqrt{b}=\sqrt{c}:1$.

25. $a+b:c+d=a-b:c-d$.

26. $a:a+b=a+c:a+b+c+d$.

27. $a+b:c+d=\sqrt{a^2+b^2}:\sqrt{c^2+d^2}$.

28. $a^3+a^2b+ab^2+b^3:a^3=c^3+c^2d+cd^2+d^3:c^3$.

29. $2a+3b:3a+4b=2c+3d:3c+4d$.

30. $2a+3c:2a-3c=8b+12d:8b-12d$.

31. $a+b+c+d:a-b+c-d=a+b-c-d:a-b-c+d$.

32. If $a:b=c:d$, and if x be a third proportional to a and b, and y a third proportional to b and c, show that the mean proportional between x and y is equal to that between c and d.

33. Solve for x, $\dfrac{\sqrt{x+7}+\sqrt{x}}{4+\sqrt{x}}=\dfrac{\sqrt{x+7}-\sqrt{x}}{4-\sqrt{x}}$.

SOLUTION

By alternation, Prin. 4,

$$\frac{\sqrt{x+7}+\sqrt{x}}{\sqrt{x+7}-\sqrt{x}}=\frac{4+\sqrt{x}}{4-\sqrt{x}}.$$

By composition and division, Prin. 8,

$$\frac{2\sqrt{x+7}}{2\sqrt{x}}=\frac{8}{2\sqrt{x}}.$$

Since the consequents are equal, the antecedents are equal.

Therefore, $2\sqrt{x+7}=8$.

Solving, $x=9$.

34. Given $\dfrac{\sqrt{x+11}+2}{\sqrt{x+11}-2}=\dfrac{\sqrt{2x+14}+2\frac{2}{3}}{\sqrt{2x+14}-2\frac{2}{3}}$, to solve for x.

Solution

By composition and division, Prin. 8,

$$\frac{2\sqrt{x+11}}{4}=\frac{2\sqrt{2x+14}}{\frac{16}{3}}.$$

Dividing both terms of each ratio by 2, Prin. 10,

$$\frac{\sqrt{x+11}}{2}=\frac{\sqrt{2x+14}}{\frac{8}{3}}.$$

Dividing the consequents by $\frac{2}{3}$, Prin. 10,

$$\frac{\sqrt{x+11}}{3}=\frac{\sqrt{2x+14}}{4}.$$

By alternation, Prin. 4, $\dfrac{\sqrt{x+11}}{\sqrt{2x+14}}=\dfrac{3}{4}.$

Squaring and applying Prin. 7,

$$\frac{x+3}{x+11}=\frac{7}{9}.$$

Solving, $x=25.$

Solve for x by the principles of proportion:

35. $\dfrac{\sqrt{x}+\sqrt{m}}{\sqrt{x}-\sqrt{m}}=\dfrac{m}{n}.$

36. $\dfrac{\sqrt{x}+\sqrt{2a}}{\sqrt{x}-\sqrt{2a}}=\dfrac{2}{1}.$

37. $\dfrac{x+\sqrt{x-1}}{x-\sqrt{x-1}}=\dfrac{13}{7}.$

38. $\dfrac{\sqrt{x+b}+\sqrt{x-b}}{\sqrt{x+b}-\sqrt{x-b}}=a.$

39. $\dfrac{\sqrt{a}+\sqrt{a-x}}{\sqrt{a}-\sqrt{a-x}}=\dfrac{1}{a}.$

40. $\dfrac{\sqrt{ax}-b}{\sqrt{ax}+b}=\dfrac{3\sqrt{ax}-2b}{3\sqrt{ax}+5b}.$

41. $\dfrac{\sqrt{a}+\sqrt{a+x}}{\sqrt{a}-\sqrt{a+x}}=\dfrac{\sqrt{b}+\sqrt{x-b}}{\sqrt{b}-\sqrt{x-b}}.$

42. $\dfrac{\sqrt{x+1}+\sqrt{x-2}}{\sqrt{x+1}-\sqrt{x-2}}=\dfrac{\sqrt{x-3}+\sqrt{x-4}}{\sqrt{x-3}-\sqrt{x-4}}.$

Problems

94. 1. Divide $35 between two men so that their shares ll be in the ratio of 3 to 4.

2. Two numbers are in the ratio of 3 to 2. If each is eased by 4, the sums will be in the ratio of 4 to 3. What the numbers?

3. Divide 16 into two parts such that their product is to sum of their squares as 3 is to 10.

4. Divide 25 into two parts such that the greater increased . is to the less decreased by 1 as 4 is to 1.

5. The sum of two numbers is 4, and the square of their is to the sum of their squares as 8 is to 5. What are the ibers?

6. Find a number that added to each of the numbers 1, 2, id 7 will give four numbers in proportion.

7. In the state of Minnesota the ratio of native-born in- itants to foreign-born recently was 5 : 2. What was the iber of each, if the total population was 1,750,000?

8. A business worth $19,000 is owned by three partners. share of one partner, $6000, is a mean proportional be- en the shares of the other two. Find the share of each.

9. What number must be added to each of the numbers 11, 2, and 5 so that the sums shall be in proportion when n in the order given?

0. Four numbers are in proportion; the difference between first and the third is $2\frac{3}{8}$; the sum of the second and the d is $6\frac{1}{8}$; the third is to the fourth as 4 : 5. Find the num- .

1. Prove that no four consecutive integers, as n, $n+1$, 2, and $n+3$, can form a proportion.

2. Prove that the ratio of an odd number to an even num- as $2m+1:2n$, cannot be equal to the ratio of another i *number to another* odd number, as $2x:2y+1$.

13. The area of the right triangle shown in Fig. 1 may be expressed either as $\frac{1}{2}\,ab$ or as $\frac{1}{2}\,ch$. Form a proportion whose terms shall be a, b, c, and h.

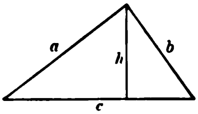


FIG. 1.

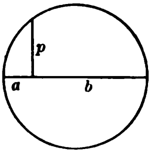


FIG. 2.

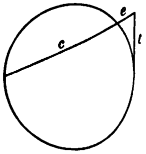


FIG. 3.

14. In Fig. 2, the perpendicular p, which is 20 feet long, is a mean proportional between a and b, the parts of the diameter, which is 50 feet long. Find the length of each part.

15. In Fig. 3, the tangent t is a mean proportional between the whole secant $c+e$, and its external part e. Find the length of t, if $e = 9\frac{3}{5}$ and $c = 50\frac{2}{5}$.

16. The strings of a musical instrument produce sound by vibrating. The relation between the number of vibrations N and N' of two strings, different only in their lengths l and l', is expressed by the proportion

$$N : N' = l' : l.$$

A c string and a g string, exactly alike except in length, vibrate 256 and 384 times per second, respectively. If the c string is 42 inches long, find the length of the g string.

17. If L and l are the lengths of two pendulums and T and t the times they take for an oscillation, then

$$T^2 : t^2 = L : l.$$

A pendulum that makes one oscillation per second is approximately 39.1 inches long. How often does a pendulum 156.4 inches long oscillate?

18. Using the proportion of exercise 17, find how many feet long a pendulum would have to be to oscillate once a minute.

VARIATION

495. Many problems and discussions in mathematics have to do with numbers some of which have *values that are continually changing* while others *remain the same* throughout the discussion. Numbers of the first kind are called **variables**; numbers of the second kind are called **constants**.

Thus, the distance of a moving train from a certain station is a variable, but the distance from one station to another is a constant.

Two variables may be so related that when one changes the other changes correspondingly.

496. One quantity or number is said to **vary directly** as another, or simply to **vary** as another, when the two depend on each other in such a manner that if one is changed the other is changed *in the same ratio.*

Thus, if a man earns a certain sum per day, the amount of wages he earns *varies* as the number of days he works.

497. The **sign of variation** is $\propto$. It is read '*varies as.*'

Thus, $x \propto y$, read 'x varies as y', is a brief way of writing the proportion

$$x : x' = y : y',$$

which x' is the value to which x is changed when y is changed to y'.

498. The expression $x \propto y$ means that if x is doubled, y is doubled, or if x is divided by a number, y is divided by the same number, etc.; that is, that the ratio of x to y is always the same, or *constant.*

If the constant ratio is represented by k, then when $x \propto y$, $= k$, or $x = ky$. Hence,

If x varies as y, x is equal to y multiplied by a constant.

499. One quantity or number varies **inversely** as another when it varies as the *reciprocal* of the *other*.

Thus, the time required to do a certain piece of work varies *inversely* as the number of men employed. For, if it takes 10 men 4 days to do a piece of work, it will take 5 men 8 days, or 1 man 40 days, to do it.

In $x \propto \frac{1}{y}$, if the constant ratio of x to $\frac{1}{y}$ is k, $\frac{x}{\frac{1}{y}} = k$, or $xy = k$. Hence,

If x varies inversely as y, their product is a constant.

500. One quantity or number varies **jointly** as two others when it varies as their product.

Thus, the amount of money a man earns varies *jointly* as the number of days he works and the sum he receives per day. For, if he should work *three* times as many days, and receive *twice* as many dollars per day, he would receive *six* times as much money.

In $x \propto yz$, if the constant ratio of x to yz is k,

$$\frac{x}{yz} = k, \text{ or } x = kyz. \quad \text{Hence,}$$

If x varies jointly as y and z, x is equal to their product multiplied by a constant.

501. One quantity or number varies **directly** as a second and **inversely** as a third when it varies *jointly* as the second and the reciprocal of the third.

Thus, the time required to dig a ditch varies *directly* as the length of the ditch and *inversely* as the number of men employed. For, if the ditch were 10 times as long and 5 times as many men were employed, it would take twice as long to dig it.

In $x \propto y \cdot \frac{1}{z}$, or $x \propto \frac{y}{z}$, if k is the constant ratio,

$$x \div \frac{y}{z} = k, \text{ or } x = k\frac{y}{z}. \quad \text{Hence,}$$

If x varies directly as y and inversely as z, x is equal to $\frac{y}{z}$ multiplied by a constant.

502. *If x varies as y when z is constant, and x varies as z en y is constant, then x varies as yz when both y and z are iable.*

Thus, the area of a triangle varies as the base when the altitude is con- it; as the altitude when the base is constant; and as the product of base and altitude when both vary.

PROOF

Since the variation of x depends upon the variations of y and z, sup- e the latter variations to take place in succession, each in turn pro- ing a corresponding variation in x.

While z remains constant, let y change to y_1, thus causing x to nge to x'.

Then, $$\frac{x}{x'} = \frac{y}{y_1}. \quad (1)$$

Now while y keeps the value y_1, let z change to z_1, thus causing x' change to x_1.

Then, $$\frac{x'}{x_1} = \frac{z}{z_1}. \quad (2)$$

Multiplying (1) by (2), $$\frac{x}{x_1} = \frac{yz}{y_1 z_1}. \quad (3)$$

$$x = \frac{x_1}{y_1 z_1} \cdot yz. \quad (4)$$

Since, if *both* changes are made, x_1, y_1, and z_1 are constants, $\frac{x_1}{y_1 z_1}$ is a stant, which may be represented by k.

Then, (4) becomes $x = kyz$.

Hence, $x \propto yz$.

Similarly, if x varies as each of three or more numbers, y, z, ⋯ when the others are constant, when all vary x varies as ir product.

That is, $x \propto yzv \cdots$.

Thus, the volume of a rectangular solid varies as the length, if the width l thickness are constant; as the width, if the length and thickness are stant; as the thickness, if the length and width are constant; as the duct of any two dimensions, if the other dimension is constant; and he product of the three dimensions, if all vary.

EXERCISES

503. 1. If x varies inversely as y, and $x = 6$ when $y = 8$, what is the value of x when $y = 12$?

SOLUTION

Since $x \propto \frac{1}{y}$, let k be the constant ratio of x to $\frac{1}{y}$.

Then, § 499, $xy = k$. (1)

Hence, when $x = 6$ and $y = 8$, $k = 6 \times 8$, or 48. (2)

Since k is constant, $k = 48$ when $y = 12$,

and (1) becomes $12x = 48$.

Therefore, when $y = 12$, $x = 4$.

2. If $x \propto \frac{y}{z}$, and if $x = 2$ when $y = 12$ and $z = 2$, what is the value of x when $y = 84$ and $z = 7$?

3. If $x \propto \frac{y}{z}$, and if $x = 60$ when $y = 24$ and $z = 2$, what is the value of y when $x = 20$ and $z = 7$?

4. If x varies jointly as y and z and inversely as the square of w, and if $x = 30$ when $y = 3$, $z = 5$, and $w = 4$, what is the value of x expressed in terms of y, z, and w?

5. If $x \propto y$ and $y \propto z$, prove that $x \propto z$.

PROOF

Since $x \propto y$ and $y \propto z$, let m represent the constant ratio of x to y, and m the constant ratio of y to z.

Then, § 498, $x = my$, (1)

and $y = nz$. (2)

Substituting nz for y in (1), $x = mnz$. (3)

Hence, since mn is constant, $x \propto z$.

6. If $x \propto \frac{1}{y}$, and $y \propto \frac{1}{z}$, prove that $x \propto z$.

7. If $x \propto y$ and $z \propto y$, prove that $(x \pm z) \propto y$.

Problems

504. 1. The volume of a cone varies jointly as its altitude d the square of the diameter of its base. When the altitude 15 and the diameter of the base is 10, the volume is 392.7. hat is the volume when the altitude is 5 and the diameter the base is 20?

SOLUTION

Let V, H, and D denote the volume, altitude, and diameter of the base, pectively, of any cone, and V' the volume of a cone whose altitude is 5 l the diameter of whose base is 20.

Since $V \propto HD^2$, or $V = kHD^2$,

l $V = 392.7$ when $H = 15$ and $D = 10$,

$$392.7 = k \times 15 \times 100. \quad (1)$$

Also, since V becomes V' when $H = 5$ and $D = 20$,

$$V' = k \times 5 \times 400. \quad (2)$$

Dividing (2) by (1), Ax. 4, $$\frac{V'}{392.7} = \frac{5 \times 400}{15 \times 100} = \frac{4}{3}. \quad (3)$$

$$\therefore V' = \tfrac{4}{3} \text{ of } 392.7 = 523.6.$$

2. The circumference of a circle varies as its diameter. If e circumference of a circle whose diameter is 1 foot is 3.1416 ›t, find the circumference of a circle 100 feet in diameter.

3. The area of a circle varies as the square of its diameter. the area of a circle whose diameter is 10 feet is 78.54 square ›t, what is the area of a circle whose diameter is 20 feet?

4. The distance a body falls from rest varies as the square the time of falling. If a stone falls 64.32 feet in 2 seconds, w far will it fall in 5 seconds?

5. The volume of a sphere varies as the cube of its diameter. the ratio of the sun's diameter to the earth's is 109.3, how ny times the volume of the earth is the volume of the sun?

6. If 10 men can do a piece of work in 20 days, how long ll it take 25 men to do it?

7. If a men can do a piece of work in b days, how many n will be required to do it in c days?

8. The area of a triangle varies jointly as its base and altitude. The area of a triangle whose base is 12 inches and whose altitude is 6 inches is 36 square inches. What is the area of a triangle whose base is 8 inches and whose altitude is 10 inches? What is the constant ratio?

9. A wrought-iron bar 1 square inch in cross section and 1 yard long weighs 10 pounds. If the weight of a uniform bar of given material varies jointly as its length and the area of its cross section, what is the weight of a wrought-iron bar 36 feet long, 4 inches wide, and 4 inches thick?

10. The weight of a beam varies jointly as the length, the area of the cross section, and the material of which it is composed. If wood is $\frac{1}{12}$ as heavy as wrought iron (see exercise 9), what is the weight of a wooden beam 24 feet long, 12 inches wide, and 12 inches thick?

11. What is the weight of a brick 2 in. $\times$ 4 in. $\times$ 8 in., if the material is $\frac{1}{4}$ as heavy as wrought iron? (For the weight of wrought iron, see exercise 9.)

12. The distances, from the fulcrum of a lever, of two weights that balance each other vary inversely as the weights. If two boys weighing 80 pounds and 90 pounds, respectively, are balanced on the ends of a board $8\frac{1}{2}$ feet long, how much of the board has each on his side of the fulcrum?

13. A water carrier carries two buckets of water suspended from the ends of a 4-foot stick that rests on his shoulder. If one bucket weighs 60 pounds and the other 100 pounds, and they balance each other, what point of the stick rests on his shoulder?

14. The horse power (H) that a solid shaft can transmit safely varies jointly as its speed in revolutions per minute (N) and the cube of its diameter. A 5-inch solid steel shaft making 150 revolutions per minute can transmit 585 horse power. How many horse power could it transmit at half this speed, if its diameter were increased 1 inch?

15. The weight of a body near the earth varies inversely as he square of its distance from the center of the earth. If the adius of the earth is 4000 miles, what would be the weight of 4-pound brick 4000 miles above the earth's surface?

16. The weight of wire of given material varies jointly as he length and the square of the diameter. If 3 miles of wire 08 of an inch in diameter weighs 288 pounds, find the weight f $\frac{1}{2}$ mile of wire .16 of an inch in diameter.

17. The illumination from a source of light varies inversely s the square of the distance. How far must a screen that is 0 feet from a lantern be moved so as to receive one fourth as nuch light?

18. The number of times a pendulum oscillates in a given ime varies inversely as the square root of its length. If a endulum 39.1 inches long oscillates once a second, what is the ength of a pendulum that oscillates twice a second? once in hree seconds?

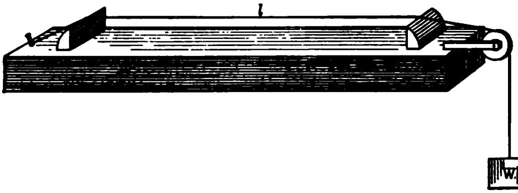
19. Three spheres of lead whose radii are 6 inches, 8 inches, nd 10 inches, respectively, are united into one. Find the adius of the resulting sphere, if the volume of a sphere varies s the cube of its radius.

20. The volume of a cone varies jointly as its altitude and he square of the diameter of its base. The altitudes of three ones, S, P, and R, are 30 ft., 10 ft., and 5 ft., respectively. he diameter of the base of P is 5 ft. and that of R is 10 ft. f the volume of S is equivalent to that of P and R combined, hat is the diameter of the base of S?

21. A boy wishes to ascertain the height of a tower. He nows that it is 31 feet 6 inches from his window to the pavement below, and that the distance through which a body falls aries as the square of the time of falling. He drops a marble rom his window and finds that it strikes the pavement in 1.4 econds. Then throwing a stone upward he observes that it akes just 3 seconds for it to descend from the top of the ower to the ground. What is the height of the tower?

505. Algebraic expression of physical laws.

If two wires exactly alike in all respects except in length (l) are stretched by equal weights, the greater number (n) of vibra-



tions per second will be made by the shorter wire. If one wire is half as long as the other, its rate of vibration will be twice as great; if $\frac{1}{3}$ as long, the rate will be 3 times as great; etc.

This result is expressed by the variation

$$n \propto \frac{1}{l}.$$

Next, experimenting with two wires alike in all respects except in diameter (d), it is found that

$$n \propto \frac{1}{d}.$$

Next, excluding all variable elements in the experiment except the stretching weight (W), it is found that

$$n \propto \sqrt{W}.$$

Finally, experimenting with wires of different materials, as steel and brass, which have different specific gravities (s),

$$n \propto \frac{1}{\sqrt{s}}.$$

Since the number of vibrations per second varies inversely as l and d, directly as the square root of W, and inversely as the square root of s, by § 502,

$$n \propto \frac{\sqrt{W}}{ld\sqrt{s}},$$

which is the expression of the law as a variation.

It is found by measuring n, l, d, W, and s in any case that the constant ratio of the first member to the second is $\sqrt{\frac{1}{\pi}}$. Hence, the law may be expressed by the *equation*

$$n = \sqrt{\frac{1}{\pi}} \cdot \frac{\sqrt{W}}{ld\sqrt{s}}.$$

EXERCISES

506. In the following use 3.1416 for the value of π, and 32.16 or 980 for the numerical value of g according as the distance unit is 1 foot or 1 centimeter. Regard g as constant.

Express by a variation, and when k is given by an equation, each of the following laws:

1. The distance (s) passed through in l seconds by a body falling freely from a state of rest varies as the square of the time. The constant ratio (k) is equal to $\frac{1}{2}g$.

2. The time required by a simple pendulum to make a complete oscillation varies as the square root of its length. $k = 2\pi \div \sqrt{g}$ is the constant ratio, at any given place.

3. The velocity (v) acquired by a body falling from a height (h) varies as the square root of the height. The constant ratio, for any given place, is $\sqrt{2g}$.

4. The quantity (Q) of water flowing from a circular orifice, of diameter (d) and under a height, or head (h), of water varies as the square of d and as the square root of h. The constant ratio, under ordinary conditions, is $k = .625 \cdot \frac{1}{4}\pi\sqrt{2g}$.

5. The intensity of a current (I) in an electric circuit varies directly as the electromotive force (E) and inversely as the resistance (R) in the circuit. The constant ratio is 1.

6. The heat loss (P) in an electric circuit varies directly as the intensity of the current (I) and the square of the resistance (R). The constant ratio is 1.

PROGRESSIONS

507. A succession of numbers, each of which after the first is derived from the preceding number or numbers according to some fixed law, is called a **series.**

The successive numbers are called the **terms** of the series. The first and last terms are called the **extremes,** and all the others, the **means.**

In the series 2, 4, 6, 8, 10, 12, 14, each term after the first is greater by 2 than the preceding term. This is the **law** of the series. Also since 1st term $= 2 \cdot 1$, 2d term $= 2 \cdot 2$, 3d term $= 2 \cdot 3$, etc., the law of the series may be expressed thus:

$$n\text{th term} = 2\,n.$$

In the series 2, 4, 8, 16, 32, 64, 128, each term after the first is twice the preceding term; or expressing the law of the series by an equation, or formula,

$$n\text{th term} = 2^n.$$

ARITHMETICAL PROGRESSIONS

508. A series, each term of which after the first is derived from the preceding by the addition of a constant number, is called an **arithmetical series,** or an **arithmetical progression.**

The number that is added to any term to produce the next is called the **common difference.**

2, 4, 6, 8, $\cdots$ and 15, 12, 9, 6, $\cdots$ are arithmetical progressions. In the first, the common difference is 2 and the series is ascending; in the second, the common difference is -3 and the series is descending.

A.P. is an abbreviation of the words *arithmetical progression.*

09. To find the *n*th, or last, term of an arithmetical series.

n the arithmetical series

$$1, 3, 5, 7, 9, 11, 13, 15, 17, 19,$$

common difference is 2, or $d = 2$. This difference enters *e* in the *second* term, for $3 = 1 + d$; *twice* in the *third* term, $5 = 1 + 2d$; *three* times in the *fourth* term, for $7 = 1 + 3d$; so on to the 10th, or last, term, which equals $1 + 9d$.

n $$a, a + d, a + 2d, a + 3d, \cdots,$$

ich is the general form of an arithmetical progression, a resenting the first term and d the common difference, erve that the coefficient of d in the expression for any n is *one less* than the number of the term.

hen, if the nth, or last, term is represented by l,

$$l = a + (n - 1)d. \tag{I}$$

OTE. — The common difference d may be either positive or negative. he A.P. 25, 23, 21, 19, 17, 15, $d = -2$.

EXERCISES

10. 1. What is the 10th term of the series 6, 9, 12, $\cdots$?

PROCESS

$a + (n - 1)d$
$6 + (10 - 1)3$
33

EXPLANATION. — Since the series 6, 9, 12, $\cdots$ is an A.P. the common difference of whose terms is 3, by substituting 6 for a, 10 for n, and 3 for d in the formula for the last term, the last term is found to be 33.

2. Find the 20th term of the series 7, 11, 15, $\cdots$.

3. Find the 16th term of the series 2, 7, 12, $\cdots$.

4. Find the 24th term of the series 1, 16, 31, $\cdots$.

5. Find the 18th term of the series 1, 8, 15, $\cdots$.

6. Find the 13th term of the series -3, 1, 5, $\cdots$.

7. *Find the* 49th term of the series 1, $1\frac{1}{3}$, $1\frac{2}{3}$, $\cdots$

8. Find the 15th term of the series 45, 43, 41, $\cdots$.

SUGGESTION. — The common difference is -2.

9. Find the 10th term of the series 5, 1, -3, $\cdots$.

10. Find the 16th term of the series a, $3a$, $5a$, $\cdots$.

11. Find the 7th term of the series $x-3y$, $x-2y$, $\cdots$.

12. A body falls $16\frac{1}{12}$ feet the first second, 3 times as far the second second, 5 times as far the third second, etc. How far will it fall during the 10th second?

511. To find the sum of *n* terms of an arithmetical series.

Let a represent the first term of an A.P., d the common difference, l the last term, n the number of terms, and s the sum of the terms.

Write the sum of n terms in the usual order and then in the reverse order, and add the two equal series; thus,

$$s = a + (a+d) + (a+2d) + (a+3d) + \cdots + l.$$

$$s = l + (l-d) + (l-2d) + (l-3d) + \cdots + a.$$

$$2s = (a+l) + (a+l) + (a+l) + (a+l) + \cdots + (a+l).$$

$$\therefore 2s = n(a+l).$$

$$s = \frac{n}{2}(a+l), \text{ or } n\left(\frac{a+l}{2}\right). \qquad \text{(II)}$$

EXERCISES

512. 1. Find the sum of 20 terms of the series 2, 5, 8, $\cdots$.

PROCESS

$$l = a + (n-1)d = 2 + (20-1)\times 3 = 59$$

$$s = n\left(\frac{a+l}{2}\right) = 20\left(\frac{2+59}{2}\right) = 610$$

Explanation. — Since the last term is not given, it is found by formula I and substituted for l in the formula for the sum.

Find the sum of:

2. 16 terms of the series 1, 5, 9, ...

3. 10 terms of the series -2, 0, 2, ...

4. 6 terms of the series 1, $3\frac{1}{2}$, 6, ...

5. 8 terms of the series a, $3a$, $5a$, ...

6. n terms of the series 1, 7, 13, ...

7. a terms of the series x, $x+2a$, ...

8. 7 terms of the series 4, 11, 18, ...

9. 10 terms of the series 1, -1, -3, ...

10. 10 terms of the series 1, $\frac{1}{2}$, 0, ...

11. How many strokes does a common clock, striking hours, ake in 12 hours?

12. A body falls $16\frac{1}{12}$ feet the first second, 3 times as far e second second, 5 times as far the third second, etc. How r will it fall in 10 seconds?

13. Thirty flower pots are arranged in a straight line 4 feet art. How far must a lady walk who, after watering each ant, returns to a well 4 feet from the first plant and in line ith the plants, if we assume that she starts at the well?

14. How long is a toboggan slide, if it takes 12 seconds for toboggan to reach the bottom by going 4 feet the first second d increasing its velocity 2 feet each second?

15. Starting from rest, a train went .18 feet the first second, t feet the next second, .90 feet the third second, and so on, aching its highest speed in 3 minutes 40 seconds. How far 1 the train go before reaching top speed?

16. In a potato race each contestant has to start from a ırk and bring back, one at a time, 8 potatoes, the first of ıich is 6 feet from the mark and each of the others 6 feet rther than the preceding. How far must each contestant *in order to finish* the race?

513. The two fundamental formulæ,

$$\text{(I) } l = a + (n-1)d \text{ and (II) } s = \frac{n}{2}(a+l),$$

contain *five* **elements**, a, d, l, n, and s. Since these formulæ are independent simultaneous equations, if they contain but two unknown elements they may be solved. Hence, if any *three* of the five elements *are known*, the other *two* may be found.

EXERCISES

514. 1. Given $d = 3$, $l = 58$, $s = 260$, to find a and n.

SOLUTION

Substituting the known values in (I) and (II), we have

$$58 = a + (n-1)3, \text{ or } a + 3n = 61; \quad (1)$$

and

$$260 = \tfrac{1}{2}n(a+58), \text{ or } an + 58n = 520. \quad (2)$$

Solving, $n = \frac{104}{3}$ or 5,

and, rejecting $n = \frac{104}{3}$, $a = 46$.

Since the number of terms must be a positive integer, fractional or negative values of n are rejected whenever they occur.

2. Given $a = 11$, $d = -2$, $s = 27$, to find the series.

SOLUTION

Substituting the known values in (I) and (II), we have

$$l = 11 + (n-1)(-2), \text{ or } l = 13 - 2n; \quad (1)$$

and

$$27 = \tfrac{1}{2}n(11+l), \text{ or } 54 = 11n + ln. \quad (2)$$

Solving, $n = 3$ or 9 and $l = 7$ or -5. (3)

Hence, the series is 11, 9, 7,

or 11, 9, 7, 5, 3, 1, -1, -3, -5.

3. How many terms are there in the series 2, 6, 10, ⋯, 66?

4. What is the sum of the series 1, 6, 11, ⋯, 61?

5. How many terms are there in the series $-1, 2, 5, \cdots$, if he sum is 221?

6. Complete the series $2, 9, 16, \cdots, 86$.

7. Complete the series $-10, -8\frac{1}{2}, -7, \cdots$ to 10 terms.

8. The sum of the series $\cdots, 22, 27, 32, \cdots$ is 714. If there e 17 terms, what are the first and last terms?

9. If $s = 113\frac{2}{3}$, $a = \frac{1}{3}$, and $d = 2$, find n.

10. What is the sum of the series $-16, -11, -6, \cdots, 34$?

11. What is the sum of the series $\cdots, -1, 3, 7, \cdots, 23$, if the ımber of terms is 16?

12. What are the extremes of the series $\cdots, 8, 10, 12, \cdots$, if $= 300$, and $n = 20$?

13. Find an A.P. of 14 terms having 10 for its 6th term, 0 r its 11th term, and 98 for the sum of the terms.

14. Find an A.P. of 15 terms such that the sum of the 5th, h, and 7th terms is 60, and that of the last three terms, 132.

From (I) and (II) derive the formula for:

15. l in terms of a, n, s.

16. s in terms of a, d, l.

17. a in terms of d, n, s.

18. d in terms of a, n, s.

19. d in terms of l, n, s.

20. n in terms of a, l, s.

515. To insert arithmetical means.

EXERCISES

1. Insert 5 arithmetical means between 1 and 31.

SOLUTION

Since there are 5 means, there must be 7 terms. Hence, in $l = a + (n - 1)d$, $l = 31$, $a = 1$, $n = 7$, and d is unknown.

Solving, $d = 5$.

Hence, 1, 6, 11, 16, 21, 26, 31 is the series.

2. Insert 9 arithmetical means between 1 and 6.
3. Insert 10 arithmetical means between 24 and 2.
4. Insert 7 arithmetical means between 10 and -14.
5. Insert 6 arithmetical means between -1 and 2.
6. Insert 14 arithmetical means between 15 and 20.
7. Insert 3 arithmetical means between $a-b$ and $a+b$.

516. If A is the arithmetical mean between a and b in the series

$$a, A, b,$$

by § 508,
$$A-a=b-A.$$

$$\therefore A=\frac{a+b}{2}.$$ That is,

PRINCIPLE. — *The arithmetical mean between two numbers is equal to half their sum.*

EXERCISES

517. Find the arithmetical mean between:

1. $\frac{2}{3}$ and $\frac{1}{2}$.
2. $a+b$ and $a-b$.
3. $(a+b)^2$ and $(a-b)^2$.
4. $\frac{x+y}{x-y}$ and $\frac{x-y}{x+y}$.
5. $1-x$ and $\frac{(1-x)^2}{1+x}$.

Problems

518. Problems in arithmetical progression involving two unknown elements commonly suggest series of the form,

$$x,\ x+y,\ x+2y,\ x+3y, \text{ etc.}$$

Frequently, however, the solution of problems is more readily accomplished by representing the series as follows:

1. When there are *three* terms, the series may be written,

$$x-y,\ x,\ x+y.$$

2. When there are *five* terms, the series may be written,

$$x-2y,\ x-y,\ x,\ x+y,\ x+2y.$$

3. When there are *four* terms, the series may be written,

$$x - 3y,\ x - y,\ x + y,\ x + 3y.$$

The sum of the terms of a series represented as above evidently contains but one unknown number.

1. The sum of three numbers in arithmetical progression is 30 and the sum of their squares is 462. What are the numbers?

SOLUTION

Let the series be $x - y,\ x,\ x + y.$

Then, $(x - y) + x + (x + y) = 30,$ (1)

and $(x - y)^2 + x^2 + (x + y)^2 = 462.$ (2)

From (1), $3x = 30;$ (3)

whence, $x = 10.$ (4)

From (2), $3x^2 + 2y^2 = 462.$ (5)

Substituting (4) in (5), $2y^2 = 162.$

Solving, $y = \pm 9.$

Forming the series from $x = 10$ and $y = \pm 9$, the terms are

1, 10, 19 or 19, 10, 1.

2. The sum of three numbers in arithmetical progression is 18, and their product is 120. What are the numbers?

3. The sum of three numbers in arithmetical progression is 21, and the sum of their squares is 155. Find the numbers.

4. There are three numbers in arithmetical progression the sum of whose squares is 93. If the third is 4 times as large as the first, what are the numbers?

5. Find the sum of the odd numbers 1 to 99, inclusive.

6. The product of the extremes of an arithmetical progression of 10 terms is 70, and the sum of the series is 95. What are the extremes?

7. Fifty-five logs are to be piled so that the top layer shall consist of 1 log, the next layer of 2 logs, the next layer of 3 logs, etc. How many logs must be placed in the bottom layer?

8. It cost Mr. Smith \$19.00 to have a well dug. If the cost of digging was \$1.50 for the first yard, \$1.75 for the second, \$2.00 for the third, etc., how deep was the well?

9. How many arithmetical means must be inserted between 4 and 25, so that the sum of the series may be 116?

10. Prove that equal multiples of the terms of an arithmetical progression are in arithmetical progression.

11. Prove that the difference of the squares of consecutive integers are in arithmetical progression, and that the common difference is 2.

12. Prove that the sum of n consecutive odd integers, beginning with 1, is n^2.

GEOMETRICAL PROGRESSIONS

519. A series of numbers each of which after the first is derived by multiplying the preceding number by some constant multiplier is called a **geometrical series**, or a **geometrical progression.**

2, 4, 8, 16, 32 and a^4, a^3, a^2, a are geometrical progressions.
In the first series the constant multiplier is 2; in the second it is $\frac{1}{a}$.
G.P. is an abbreviation of the words *geometrical progression.*

520. The constant multiplier is called the **ratio.**

It is evident that the terms of a geometrical progression *increase* or *decrease* numerically according as the ratio is numerically *greater* or *less* than 1.

521. **To find the nth, or last, term of a geometrical series.**

Let a represent the first term of a G.P., r the ratio, n the number of terms, and l the last, or nth, term.

Then, the series is a, ar, ar^2, ar^3, ar^4, $\cdots$.

Observe that the exponent of r is one less than the number *of the term;* that is,

$$l = ar^{n-1}. \qquad (I)$$

EXERCISES

522. 1. Find the 9th term of the series 1, 3, 9, ⋯.

PROCESS

$l = ar^{n-1}$

$= 1 \times 3^8$

$= 6561$

EXPLANATION. — In this exercise $a = 1$, $r = 3$, and $n = 9$.

Substituting these values in the formula for l, the last term is found to be 6561.

2. Find the 10th term of the series 1, 2, 4, ⋯.

3. Find the 8th term of the series $\frac{1}{4}$, $\frac{1}{2}$, 1, ⋯.

4. Find the 9th term of the series 6, 12, 24, ⋯.

5. Find the 11th term of the series $\frac{1}{2}$, 1, 2, ⋯.

6. Find the 7th term of the series 2, 6, 18, ⋯.

7. Find the 6th term of the series 4, 20, 100, ⋯.

8. Find the 6th term of the series 6, 18, 54, ⋯.

9. Find the 10th term of the series 1, $\frac{1}{2}$, $\frac{1}{4}$, ⋯.

10. Find the 10th term of the series 1, $\frac{2}{3}$, $\frac{4}{9}$, ⋯.

11. Find the 8th term of the series $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{9}$, ⋯.

12. Find the 11th term of the series $a^{19}b$, $a^{18}b^2$, ⋯.

13. Find the nth term of the series 2, $\sqrt{2}$, 1, ⋯.

14. If a man begins business with a capital of \$2000 and doubles it every year for 6 years, how much is his capital at the end of the sixth year?

15. The population of the United States was 76.3 millions in 1900. If it doubles itself every 25 years, what will it be in the year 2000?

16. A man's salary was raised $\frac{1}{4}$ every year for 5 years. If his salary was \$512 the first year, what was it the sixth year?

17. The population of a city, which at a certain time was 20,736, increased in geometrical progression 25 % each decade. What was the population at the end of 40 years?

18. A man who wanted 10 bushels of wheat thought \$1 a bushel too high a price; but he agreed to pay 2 cents for the first bushel, 4 cents for the second, 8 cents for the third, and so on. How much did the last bushel cost him?

19. The machinery in a manufacturing establishment is valued at \$20,000. If its value depreciates each year to the extent of 10% of its value at the beginning of that year, how much will the machinery be worth at the end of 5 years?

20. From a grain of corn there grew a stalk that produced an ear of 150 grains. These grains were planted, and each produced an ear of 150 grains. This process was repeated until there were 4 harvestings. If 75 ears of corn make 1 bushel, how many bushels were there the fourth year?

523. A series consisting of a limited number of terms is called a **finite series.**

524. A series consisting of an unlimited number of terms is called an **infinite series.**

525. To find the sum of a finite geometrical series.

Let a represent the first term, r the ratio, n the number of terms, l the nth, or last, term, and s the sum of the terms.

Then, $$s = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}. \quad (1)$$

$(1) \times r$, $$rs = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n. \quad (2)$$

$(2)-(1)$, $s(r-1) = ar^n - a.$

$$\therefore\ s = \frac{ar^n - a}{r-1}. \quad \text{(II)}$$

But, since $ar^{n-1} = l$, $\quad ar^n = rl.$

Substituting rl for ar^n in (II),

$$s = \frac{rl - a}{r-1}, \text{ or } \frac{a - rl}{1-r}. \quad \text{(III)}$$

EXERCISES

526. 1. Find the sum of 6 terms of the series 3, 9, 27, $\cdots$.

PROCESS

$$=\frac{ar^n-a}{r-1}$$

$$=\frac{3\times 3^6-3}{3-1}=1092$$

EXPLANATION. — Since the first term a, the ratio r, and the number of terms n, are known, formula II, which gives the sum in terms of a, r, and n, is used.

2. Find the sum of 8 terms of the series 1, 2, 4, $\cdots$.

3. Find the sum of 8 terms of the series 1, $\frac{1}{2}$, $\frac{1}{4}$, $\cdots$.

4. Find the sum of 10 terms of the series 1, $1\frac{1}{2}$, $2\frac{1}{4}$, $\cdots$.

5. Find the sum of 7 terms of the series 2, $-\frac{2}{3}$, $\frac{2}{9}$, $\cdots$.

6. Find the sum of 12 terms of the series $-\frac{1}{2}$, $\frac{1}{4}$, $-\frac{1}{8}$, $\cdots$.

7. Find the sum of 7 terms of the series 1, $2x$, $4x^2$, $\cdots$.

8. Find the sum of 7 terms of the series 1, $-2x$, $4x^2$, $\cdots$.

9. Find the sum of n terms of the series 1, x^2, x^4, $\cdots$.

10. Find the sum of n terms of the series 1, 2, 4, $\cdots$.

11. Find the sum of n terms of the series 1, $\frac{1}{3}$, $\frac{1}{9}$, $\cdots$.

12. The extremes of a geometrical series are 1 and 729, and ιe ratio is 3. What is the sum of the series?

13. What is the sum of the series 3, 6, 12, $\cdots$, 192?

14. What is the sum of the series 7, $\cdots$, -56, 112, -224?

527. To find the sum of an infinite geometrical series.

If the ratio r is numerically less than 1, it is evident that ıe successive terms of a geometrical series become numeric- .ly less and less. Hence, in an infinite decreasing geomet- cal series, the nth term l, or ar^{n-1}, can be made less than any *ıssignable number,* though not absolutely equal to zero.

Formula (III), page 404, may be written,

$$s = \frac{a}{1-r} - \frac{rl}{1-r}.$$

Since by taking enough terms l and, consequently, rl can be made less than any assignable number, the second fraction may be neglected.

Hence, the formula for the sum of an infinite decreasing geometrical series is

$$s = \frac{a}{1-r}. \tag{IV}$$

EXERCISES

528. 1. Find the sum of the series $1, \frac{1}{10}, \frac{1}{100}, \cdots$.

SOLUTION

Substituting 1 for a and $\frac{1}{10}$ for r in (IV),

$$s = \frac{1}{1-\frac{1}{10}} = \frac{1}{\frac{9}{10}} = \frac{10}{9}.$$

Find the value of:

2. $1 + \frac{1}{2} + \frac{1}{4} + \cdots$.
3. $3 + \frac{3}{4} + \frac{3}{16} + \cdots$.
4. $1 - \frac{1}{3} + \frac{1}{9} - \cdots$.
5. $-4 - 1 - \frac{1}{4} - \cdots$.
6. $-2 + \frac{2}{5} - \frac{2}{25} + \cdots$.
7. $100 - 10 + 1 - \cdots$.
8. $1 + x + x^2 + x^3 + \cdots$, when $x = .9$.
9. $1 - x + x^2 - x^3 + \cdots$, when $x = \frac{3}{8}$.

10. Find the value of the repeating decimal $.185185185\cdots$.

SOLUTION

Since $.185185185\cdots = .185 + .000185 + .000000185 + \cdots$, $a = .185$ and $r = .001$.

Substituting in (IV), $.185185185\cdots = s = \dfrac{.185}{1-.001} = \dfrac{5}{27}.$

Find the the value of:

11. $.407407\cdots$.
12. $.363636\cdots$.
13. $1.94444\cdots$.
14. $.020303\cdots$.
15. $.007007\cdots$.
16. $5.032828\cdots$.

529. To insert geometrical means between two terms.

EXERCISES

1. Insert 3 geometrical means between 2 and 162.

PROCESS	
$l = ar^{n-1}$	EXPLANATION. — Since there are three means, there are five terms, and $n - 1 = 4$. Solving for r and neglecting imaginary values, $r = \pm 3$.
$162 = 2r^4$	Therefore, the series is either 2, 6, 18, 54, 162 or 2, -6, 18, -54, 162.
$r = \pm 3$	

2. Insert 3 geometrical means between 1 and 625.
3. Insert 5 geometrical means between $4\frac{1}{2}$ and $\frac{2048}{81}$.
4. Insert 4 geometrical means between $\frac{343}{16}$ and $\frac{64}{49}$.
5. Insert 4 geometrical means between 5120 and 5.
6. Insert 4 geometrical means between $4\sqrt{2}$ and 1.
7. Insert 5 geometrical means between a^6 and b^6.
8. Insert 4 geometrical means between x and $-y$.

530. If G is the geometrical mean between a and b, in the eries

$$a, G, b,$$

'y § 519,

$$\frac{G}{a} = \frac{b}{G}.$$

$$G = \pm\sqrt{ab}.$$ That is,

PRINCIPLE. — *The geometrical mean between two numbers is qual to the square root of their product.*

Observe that the *geometrical mean* between two numbers is also their *ιean proportional.*

EXERCISES

531. Find the geometrical mean between:

1. 8 and 50.
2. $\frac{1}{2}$ and $3\frac{5}{9}$.
3. $1\frac{11}{16}$ and $\frac{3}{4}$.
4. $(a+b)^2$ and $(a-b)^2$.
5. $\dfrac{a^2+ab}{a^2-ab}$ and $\dfrac{ab+b^2}{ab-b^2}$.

532. Since formula I with formula II, or III, which is equivalent to II, forms a system of two independent simultaneous equations containing five elements, if *three* elements *are known*, the other *two* may be found by elimination.

NOTE. — Solving for n, since it is an exponent, requires a knowledge of logarithms (§§ 558–598), except in cases where its value may be determined by inspection. Only such cases are given in this chapter.

Problems

533. 1. Given r, l, and s, to find a.

2. The ratio of a geometrical progression is 5, the last term is 625, and the sum is 775. What is the first term?

3. The ratio of a geometrical progression is $\frac{1}{10}$, the sum is $\frac{1}{3}$, and the series is infinite. What is the first term?

4. Find l in terms of a, r, and s.

5. Find the last term of the series 5, 10, 20, $\cdots$, the sum of whose terms is 155.

6. If $\frac{1}{8} + \frac{1}{8}\sqrt{2} + \frac{1}{4} + \cdots = 1\frac{7}{8}(1 + \sqrt{2})$, what is the last term, and the number of terms?

7. Deduce the formula for r in terms of a, l, and s.

8. If the sum of the geometrical progression 32, $\cdots$, 243 is 665, what is the ratio? Write the series.

9. The sum of a geometrical progression is 700 greater than the first term and 525 greater than the last term. What is the ratio? If the first term is 81, what is the progression?

10. Deduce the formula for r in terms of a, n, and l.

11. The first term of a geometrical progression is 3, the last term is 729, and the number of terms is 6. What is the ratio? Write the series.

12. Find l in terms of r, n, and s.

13. The velocity of a sled at the bottom of a hill is 100 *feet per second*. How far will it go on the level, if its velocity *decreases each* second $\frac{1}{5}$ of that of the previous second?

14. Under normal conditions the members of a certain species of bacteria reproduce by division (each individual into two) every half hour. If no hindrance is offered, how many bacteria will a single individual produce in 8 hours?

15. A ball thrown vertically into the air 100 feet falls and rebounds 40 feet the first time, 16 feet the second time, and so on. What is the whole distance through which the ball will have passed when it finally comes to rest?

16. Show that the amount of $1 for 1, 2, 3, 4, 5 years at compound interest varies in geometrical progression.

17. Show that equal multiples of numbers in geometrical progression are also in geometrical progression.

18. The sum of three numbers in geometrical progression is 19, and the sum of their squares is 133. Find the numbers.

SUGGESTION. — When there are but three terms in the series, they may be represented by x^2, xy, y^2, or by x, $\sqrt{xy}$, y.

19. The product of three numbers in geometrical progression is 8, and the sum of their squares is 21. What are the three numbers?

20. The sum of the first and second of four numbers in geometrical progression is 15, and the sum of the third and fourth is 60. What are the numbers?

SUGGESTION. — Four unknown numbers in geometrical progression may be represented by $\frac{x^2}{y}$, x, y, $\frac{y^2}{x}$.

21. From a cask of vinegar $\frac{1}{3}$ was drawn off and the cask was filled by pouring in water. Show that if this is done 6 times, the contents of the cask will be more than $\frac{9}{10}$ water.

22. If the quantity, and correspondingly the pressure, of the air in the receiver of an air pump is diminished by $\frac{1}{10}$ of itself at each stroke of the piston, and if the initial pressure is 14.7 pounds per square inch, find, to the nearest tenth of a pound, what the pressure will be after 6 strokes.

23. A man bought a farm for \$5000, agreeing to pay principal and interest in five equal annual installments. Find the annual payment, interest at 6 %.

SOLUTION

By the conditions of the problem the equal payments are to include the interest accrued at the end of each year plus a portion of the principal.

The principal for the second year will be less than the principal for the first year by the portion of the principal paid at the end of the first year; therefore, the interest to be paid at the second payment will be less than the interest paid at the first payment by the interest for 1 year upon the first portion of the principal paid, or 6 % of the portion of the principal paid the first year.

Since the payments are to be *equal*, the portion of the principal to be paid at the second payment must be as much *more* than the portion paid at the first payment as the interest is *less* than the interest paid at the first payment; that is, it must be 6 % more than, or 1.06 of, the portion of the principal first paid.

By reasoning in the same way regarding subsequent payments, the third portion of the principal paid will be found to be 1.06 of the second, the fourth 1.06 of the third, and the fifth 1.06 of the fourth; that is, the portions of the principal paid form a G.P., in which $r = 1.06$, $n = 5$ (the number of payments), and $s = \$5000$. We desire to find a.

Substituting the known values in (I) and (III), we have

$$l = a\,1.06^{5-1}, \text{ or } l = 1.06^4\,a; \tag{1}$$

and
$$\$5000 = \frac{1.06\,l - a}{1.06 - 1}, \text{ or } l = \frac{a + \$5000 \times .06}{1.06}. \tag{2}$$

Eliminating l from (1) and (2) and solving for a, we have

$$a = \frac{\$5000 \times .06}{1.06^5 - 1} = \$886.98+.$$

That is, the first portion of the principal paid $= \$886.98$; but the first year's interest $= 6\%$ of \$5000, or \$300; hence, the entire first payment $= \$886.98 + \$300 = \$1186.98$, which is also each annual payment.

24. A man borrowed \$1500, agreeing to pay principal and interest at 6 % in four equal annual installments. Find the sum to be paid each year.

25. A father bequeathed to his son \$10,000, the bequest and interest at 4 % to be paid in six equal annual installments. Find the annual payment.

INTERPRETATION OF RESULTS

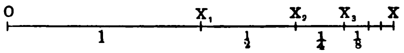
534. A number that has the same value throughout a discussion is called **a constant.**

Arithmetical numbers are constants. A literal number is constant in a discussion, if it keeps the same value throughout that discussion.

535. A number that under the conditions imposed upon it may have a series of different values is called a **variable.**

The numbers .3, .33, .333, .3333, . . . are successive values of a variable approaching in value the constant $\frac{1}{3}$.

536. When a variable takes a series of values that approach nearer and nearer a given constant without becoming equal to it, so that by taking a sufficient number of steps the difference between the variable and the constant can be made numerically less than any conceivable number however small, the constant is called the **limit** of the variable, and the variable is said to **approach its limit.**



This figure represents graphically a variable x approaching its limit $OX = 2$. The first value is $OX_1 = 1$; the second is $OX_2 = 1\frac{1}{2}$; the third is $OX_3 = 1\frac{3}{4}$; etc.

At each step the difference between the variable and its limit is diminished by half of itself. Consequently, by taking a sufficient number of steps this difference may become less than any number, however small, that may be assigned.

537. A variable that may become numerically greater than any assignable *number* is said to be **infinite.**

The symbol of an infinite number is ∞.

538. A variable that may become numerically less than any assignable number is said to be **infinitesimal.**

An infinitesimal is a variable whose limit is zero.

The character 0 is used as a symbol for an *infinitesimal number* as well as for *absolute zero,* which is the result obtained by subtracting a number from itself.

539. A number that cannot become either infinite or infinitesimal is said to be **finite.**

THE FORMS $a \times 0, \frac{0}{a}, \frac{a}{0}, \frac{a}{\infty}, \frac{0}{0}, \frac{\infty}{\infty}$

540. The results of algebraic processes may appear in the forms, $a \times 0, \frac{0}{a}, \frac{a}{0}, \frac{a}{\infty}, \frac{0}{0}, \frac{\infty}{\infty}$, etc., which are arithmetically meaningless; consequently, it becomes important to interpret the meaning of such forms.

541. Interpretation of $a \times 0$.

1. Let 0 represent *absolute zero,* defined by the identity,

$$0 = n - n. \qquad (1)$$

Multiplying $a = a$ by (1), member by member, Ax. 3, we have

$$a \times 0 = a(n - n)$$
$$= an - an$$

by def. of zero, $\quad = 0.$ That is,

Any finite number multiplied by zero is equal to zero.

2. Let 0 represent an *infinitesimal,* as the variable whose successive values are 1, .1, .01, .001, ⋯.

Then, the successive values of $a \times 0$ are (§ 81)

$$a, .1\,a, .01\,a, .001\,a, \cdots. \quad \text{Hence,}$$

$a \times 0$ is a variable whose *limit* is absolute zero. That is,

Any finite number multiplied by an infinitesimal number is equal to an infinitesimal number.

542. Interpretation of $\frac{0}{a}$.

1. Let 0 represent *absolute zero*, defined by the identity,

$$0 = n - n.$$

Dividing by a, $\qquad \frac{0}{a} = \frac{n}{a} - \frac{n}{a};$

but by def. of zero, $\qquad \frac{n}{a} - \frac{n}{a} = 0.$

Hence, Ax. 5, $\qquad \frac{0}{a} = 0.$ That is,

Zero divided by any finite number is equal to zero.

2. Let 0 represent an *infinitesimal*, as the variable whose successive values are 1, .1, .01, .001, ⋯.

Then, the successive values of $\frac{0}{a}$ are $\frac{1}{a}$, $\frac{.1}{a}$, $\frac{.01}{a}$, $\frac{.001}{a}$, ⋯;

whence, $\frac{0}{a}$ is a variable whose *limit* is absolute zero.

Hence,

Any infinitesimal number divided by a finite number is equal to an infinitesimal number.

543. Interpretation of $\frac{a}{0}$.

The successive values of the fractions, $\frac{1}{2}$, $\frac{1}{.2}$, $\frac{1}{.02}$, $\frac{1}{.002}$, etc., are .5, 5, 50, 500, etc., and they continually increase as the denominators decrease.

In general, if the numerator of the fraction $\frac{a}{x}$ is constant while the denominator decreases regularly until it becomes numerically less than any assignable number, the quotient will increase regularly and become numerically *greater* than any assignable number.

$$\therefore \frac{a}{0} = \infty. \quad \text{That is,}$$

If a finite number is divided by an infinitesimal number, the quotient will be an infinite number.

544. Interpretation of $\frac{a}{\infty}$.

The successive values of the fractions, $\frac{1}{2}$, $\frac{1}{20}$, $\frac{1}{200}$, $\frac{1}{2000}$, etc., are .5, .05, .005, .0005, etc., and they continually decrease as the denominators increase.

In general, if the numerator of the fraction $\frac{a}{x}$ is constant while the denominator increases regularly until it becomes numerically greater than any assignable number, the quotient will decrease regularly and become numerically *less* than any assignable number.

$$\therefore \frac{a}{\infty} = 0. \quad \text{That is,}$$

If a finite number is divided by an infinite number, the quotient will be an infinitesimal number.

545. Interpretation of $\frac{0}{0}$.

Let 0 represent absolute zero.

Then, if a is *any* finite number, § 541,

$$a \times 0 = 0;$$

whence,

$$\frac{0}{0} = a. \quad \text{That is,}$$

When 0 represents absolute zero, $\frac{0}{0}$ is the symbol of an indeterminate number.

546. Interpretation of $\frac{\infty}{\infty}$.

Let a represent *any finite* number and x any number whatever.

Then,

$$\frac{\frac{a}{x}}{\frac{1}{x}} = \frac{a}{x} \cdot \frac{x}{1} = a. \tag{1}$$

If x decreases regularly until it becomes numerically less than any assignable number (§ 543), $\frac{a}{x}$ and $\frac{1}{x}$ each become ∞.

Consequently, (1) becomes $\frac{\infty}{\infty} = a$, any finite number.

Hence, $\frac{\infty}{\infty}$ *is the symbol of an indeterminate number.*

547. Since (§ 543) $\frac{a}{0}$ is infinite and (§ 545) $\frac{0}{0}$ is indeterminate, it is seen that axiom 4 (§ 68) is not applicable when the divisor is 0; that is, *it is not allowable to divide by absolute zero.*

The student may point out the inadmissible step or fallacy in:

$$7x - 35 = 3x - 15,$$

$$7(x-5) = 3(x-5).$$

$$\therefore 7 = 3.$$

SUGGESTION. — Solve the equation to find what divisor has been used.

548. Fractions indeterminate in form.

Some fractions, for certain values of the variable involved, give the result $\frac{0}{0}$, which, however, is indeterminate *only in form,* because a definite value for the fraction may often be found.

For example, when $x = 1$, by substituting directly, $\frac{x^2-1}{x-1} = \frac{0}{0}$.

Though $\frac{x^2-1}{x-1} = \frac{(x+1)(x-1)}{x-1} = x+1$, it is not allowable to perform this operation in finding the value of the fraction when $x = 1$, that is, when $x - 1 = 0$, for (§ 547) it is not allowable to divide by absolute zero. However, since the value of $\frac{x^2-1}{x-1}$ is always the same as the value of $x+1$ so long as $x \neq 1$, let x approach 1 as a limit.

But (§ 536) x cannot become 1, and it is allowable to divide by $x - 1$.

Now as x approaches 1 as a limit, $\frac{x^2-1}{x-1}$ approaches $x+1$, or 2, as a limit, and so 2 is called the *value* of the fraction. That is,

The **value** of such a fraction for any given value of the variable involved is the *limit* that the fraction approaches as the *variable approaches* the given value as its limit.

THE BINOMIAL THEOREM

549. The Binomial Theorem derives a formula by means of which any indicated power of a binomial may be expanded into a series.

POSITIVE INTEGRAL EXPONENTS

550. By actual multiplication,

$$(a+x)^2 = a^2 + 2\,ax + x^2.$$

$$(a+x)^3 = a^3 + 3\,a^2x + 3\,ax^2 + x^3.$$

$$(a+x)^4 = a^4 + 4\,a^3x + 6\,a^2x^2 + 4\,ax^3 + x^4.$$

These powers of $(a+x)$ may be written, respectively:

$$(a+x)^2 = a^2 + 2\,ax + \frac{2\cdot 1}{1\cdot 2}x^2.$$

$$(a+x)^3 = a^3 + 3\,a^2x + \frac{3\cdot 2}{1\cdot 2}ax^2 + \frac{3\cdot 2\cdot 1}{1\cdot 2\cdot 3}x^3.$$

$$(a+x)^4 = a^4 + 4\,a^3x + \frac{4\cdot 3}{1\cdot 2}a^2x^2 + \frac{4\cdot 3\cdot 2}{1\cdot 2\cdot 3}ax^3 + \frac{4\cdot 3\cdot 2\cdot 1}{1\cdot 2\cdot 3\cdot 4}x^4.$$

If the law of development revealed in the above is assumed to apply to the expansion of any power of any binomial, as the nth power of $(a+x)$, the result is

$$(a+x)^n$$
$$= a^n + na^{n-1}x + \frac{n(n-1)}{1\cdot 2}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}x^3 + \cdots. \quad \text{(I)}$$

From formula (I) it is evident that in any term:

1. The exponent of x is 1 less than the number of the term. *Hence, the* exponent of x in the $(r+1)$th term is r.

2. The exponent of a is n minus the exponent of x.

Hence, the exponent of a in the $(r+1)$th term is $n-r$.

3. The number of factors in the numerator and in the denominator of any coefficient is 1 less than the number of the term.

Hence, the coefficient of the $(r+1)$th term has r factors in the numerator and r factors in the denominator.

Therefore, the $(r+1)$**th**, or **general, term,** is

$$\frac{n(n-1)(n-2)\cdots \text{ to } r \text{ factors}}{1\cdot 2\cdot 3\cdots \text{ to } r \text{ factors}}a^{n-r}x^r. \qquad (1)$$

When there are two factors in the numerator, the *last* is $n-1$; when there are three factors, $n-2$; when there are four factors, $n-3$, etc. Therefore, when there are r factors, the last is $n-(r-1)$, or $n-r+1$. Hence, (1) may be written

$$\frac{n(n-1)(n-2)\cdots(n-r+1)}{1\cdot 2\cdot 3\cdots r}a^{n-r}x^r. \qquad (2)$$

Therefore, the full form of (I) is

$$(a+x)^n$$
$$=a^n+na^{n-1}x+\frac{n(n-1)}{1\cdot 2}a^{n-2}x^2+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}x^3+\cdots$$
$$+\frac{n(n-1)(n-2)\cdots(n-r+1)}{1\cdot 2\cdot 3\cdots r}a^{n-r}x^r+\cdots+x^n.$$

This is called the **binomial formula.** It will now be proved to be true for positive integral exponents.

551. Since it has already been proved, by actual multiplication (§ 550), that the binomial formula is true for the *second, third,* and *fourth* powers of a binomial, it remains to discover whether it is true for powers higher than the fourth.

If the binomial theorem, when *assumed* to be true for the nth power, can be *proved* to be true for the $(n+1)$th power, since it is known to be true when the nth power is the *fourth* power, it will then have been proved to be true for the *fifth* power; also for the *sixth* power, being true for the fifth power; *and in like manner* for each succeeding power.

Therefore, it remains to prove that if (I) is true for the nth power, it will hold true for the $(n+1)$th power.

The $(n+1)$th power of $(a+x)$ may be obtained from the nth power by multiplying both members of (I) by $(a+x)$. Then, we have

$$
\begin{array}{l|l|l|l}
(a+x)^{n+1} & & & \\
=a^{n+1}+n & a^n x + \frac{n(n-1)}{1\cdot 2} & a^{n-1}x^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3} & a^{n-2}x^3+\cdots \\
\quad +1 & a^n x + \quad\quad n & a^{n-1}x^2 + \quad\quad \frac{n(n-1)}{1\cdot 2} & a^{n-2}x^3+\cdots.
\end{array}
$$

Collecting the coefficients of like powers of a and x, we have

Coefficient of $a^n x \quad = n+1.$

$$\text{Coefficient of } a^{n-1}x^2 = \frac{n(n-1)}{1\cdot 2}+n$$

$$= \frac{n^2-n+2n}{1\cdot 2} = \frac{(n+1)n}{1\cdot 2}.$$

$$\text{Coefficient of } a^{n-2}x^3 = \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}+\frac{n(n-1)}{1\cdot 2}$$

$$= \frac{n^3-n}{1\cdot 2\cdot 3}$$

$$= \frac{(n+1)\,n\,(n-1)}{1\cdot 2\cdot 3}.$$

$$\therefore (a+x)^{n+1} = a^{n+1}+(n+1)a^n x+\frac{(n+1)n}{1\cdot 2}a^{n-1}x^2$$

$$+\frac{(n+1)\,n\,(n-1)}{1\cdot 2\cdot 3}a^{n-2}x^3+\cdots. \quad \text{(II)}$$

Upon comparison it may be seen that (II) and (I) have the same form, $n+1$ in one taking the place of n in the other. That is, (II) and (I) express the same law of formation.

Therefore, if the formula is true for the nth power, it holds *true for* the $(n+1)$th power.

By actual multiplication (§ 550) the formula is known *to be true* for the *fourth* power. Consequently, it is true for

the *fifth* power; and then being true for the *fifth* power, it is true for the *sixth* power; and so on for each succeeding power.

Hence, the binomial formula is true for *any positive integral exponent.*

This proof is known as a proof by **mathematical induction.**

552. If $-x$ is substituted for x in (I), the terms that contain the odd powers of $-x$ will be negative, and those that contain the even powers will be positive. Therefore,

$$(a-x)^n = a^n - na^{n-1}x + \frac{n(n-1)}{1\cdot 2}a^{n-2}x^2 - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}x^3 \cdots. \quad \text{(III)}$$

If $a=1$, (I) becomes

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1\cdot 2}x^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3 + \cdots. \quad \text{(IV)}$$

553. From (I) it is seen that the last factor in the numerator of the coefficient is n for the 2d term, $n-1$ for the 3d term, $n-2$ for the 4th term, $n-(n-2)$, or 2, for the nth term, and $n-(n-1)$, or 1, for the $(n+1)$th term; and that the coefficient of the $(n+2)$th term, and of each succeeding term, contains the factor $n-n$, or 0, and therefore reduces to 0. Hence,

When n is a positive integer, the series formed by expanding $(a+x)^n$ is finite and has $n+1$ terms.

554. By formula (I) when n is a positive integer,

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{1\cdot 2}a^{n-2}x^2 + \cdots + \frac{n(n-1)\cdots 2\cdot 1}{1\cdot 2\cdots(n-1)n}x^n.$$

$$(x+a)^n = x^n + nx^{n-1}a + \frac{n(n-1)}{1\cdot 2}x^{n-2}a^2 + \cdots + \frac{n(n-1)\cdots 2\cdot 1}{1\cdot 2\cdots(n-1)n}a^n.$$

A comparison of the two series shows that:

The coefficients of the latter half of the expansion of $(a+x)^n$, when n is a positive integer, are the same as those of the first half, written in the reverse order.

EXERCISES

555. 1. Expand $(3a - 2b)^4$.

Solution. — **Substituting $3a$ for a, $2b$ for x, and 4 for n in (III),**

$$(3a - 2b)^4 = (3a)^4 - 4(3a)^3(2b) + \frac{4 \cdot 3}{1 \cdot 2}(3a)^2(2b)^2 - \frac{4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3}(3a)(2b)^3 + \frac{4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4}(2b)^4$$
$$= 81a^4 - 216a^3b + 216a^2b^2 - 96ab^3 + 16b^4.$$

2. Expand $\left(\frac{b}{2} + bx\right)^5$.

Suggestion. — **Since** $\left(\frac{b}{2} + bx\right)^5 = \left[\frac{b}{2}(1 + 2x)\right]^5 = \frac{b^5}{32}(1 + 2x)^5$, $(1 + 2x)^5$ **may be expanded by (IV), and the result multiplied by** $\frac{b^5}{32}$.

Expand:

3. $(b - n)^7$.
4. $(1 + a^{-1})^4$.
5. $(2 - 3x)^6$.
6. $(x^2 - x)^8$.
7. $(x + x^{-1})^6$.
8. $(2a + \sqrt{x})^3$.
9. $(a + a\sqrt{a})^4$.
10. $\left(1 + \frac{2}{x^2}\right)^5$.
11. $\left(\frac{a}{x} - \frac{x}{a}\right)^5$.
12. $\left(\frac{1}{x} - \frac{a}{y}\right)^8$.
13. $(\sqrt[3]{a^2} + \sqrt[4]{b^3})^3$.
14. $(2\sqrt{2} - \sqrt[3]{3})^6$.
15. $\left(\sqrt{2} + \frac{1}{\sqrt{x}}\right)^3$.
16. $(x^{\frac{n-1}{n}} - x^{\frac{1}{n}})^4$.
17. $(ax^{-2} - b\sqrt{x})^6$.
18. $\left(\frac{\sqrt{a}}{\sqrt[3]{b}} - \frac{\sqrt{b}}{\sqrt{a^3}}\right)^6$.
19. $\left(\frac{x}{y}\sqrt{\frac{x}{y}} + \frac{2}{3}\sqrt{\frac{2}{3}}\right)^3$.

556. To find any term of the expansion of $(a + x)^n$.

Any term of the expansion of a power of a binomial may be obtained by substitution in (1) or (2), § 550.

In the expansion of a power of the difference of two numbers $(a - x)^n$, *since the* exponent of x in the $(r + 1)$th term is r, the sign of the general *term is* $+$ *if* r is *even*, and $-$ if r is odd.

EXERCISES

557. 1. Find the 12th term of $(a-b)^{14}$.

SOLUTION

$$12\text{th term} = \frac{14\cdot 13\cdot 12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9\cdot 10\cdot 11}\, a^3(-b)^{11}$$

$$= -\frac{14\cdot 13\cdot 12}{1\cdot 2\cdot 3}\, a^3b^{11} = -364\, a^3b^{11}.$$

Or, since there are 15 terms, the coefficient of the 12th term, or the 4th term from the end, is equal to that of the 4th term from the beginning.

$$\therefore\ 12\text{th term} = -\frac{14\cdot 13\cdot 12}{1\cdot 2\cdot 3}\, a^3b^{11} = -364\, a^3b^{11}$$

Without actually expanding, find the:

2. 4th term of $(a+2)^{10}$.
3. 8th term of $(x-y)^{11}$.
4. 5th term of $(x-2y)^{12}$.
5. 20th term of $(1+x)^{24}$.
6. 18th term of $(1-2x)^{20}$.
7. 13th term of $(a^2-a^{-2})^{15}$.
8. Find the middle term of $(a+3b)^6$.
9. Find the 6th term of $\left(x+\dfrac{1}{x}\right)^{10}$.
10. Find the middle term of $\left(\dfrac{x}{y}-\dfrac{y}{x}\right)^8$.
11. Find the two middle terms of $\left(\dfrac{a}{b}-\dfrac{b}{a}\right)^9$.
12. In the expansion of $(x^2+x)^{11}$, find the term containing x^{15}.

SOLUTION. — Since $(x^2+x)^{11} = \left[x^2\left(1+\frac{1}{x}\right)\right]^{11} = x^{22}\left(1+\frac{1}{x}\right)^{11}$, every term of the series expanded from $\left(1+\frac{1}{x}\right)^{11}$ will be multiplied by x^{22}.

Hence, the term sought is that which contains $\left(\frac{1}{x}\right)^7$, or $\frac{1^7}{x^7}$; that is, the $(7+1)$th, or 8th term.

$$8\text{th term} = x^{22}\frac{11\cdot 10\cdot 9\cdot 8}{1\cdot 2\cdot 3\cdot 4}\left(\frac{1}{x}\right)^7 = 330\, x^{15}.$$

13. Find the coefficient of a^9 in the expansion of $(a^3+a)^5$.
14. Find the term containing $a^{18}b^9$ in the expansion of $(a-b)^{27}$ *and obtain a simple* expression for it when $b = 15^{-\frac{1}{9}}(143^{\frac{2}{9}}a^{4})^{-\frac{1}{2}}$.

The binomial formula is true for the expansion of a binomial when the exponent is negative or fractional, provided the first term of the binomial is numerically greater than the second. In such cases the expansion is an infinite series. For a more extended treatment of this subject, see the author's *Advanced Algebra*.

15. Expand $(1-y)^{-1}$ and find its $(r+1)$th term.

SOLUTION. — Substituting 1 for a, y for x, and -1 for n in (III),

$$(1-y)^{-1} = 1^{-1} - (-1)1^{-2}y + \frac{-1(-2)}{1\cdot 2}1^{-3}y^2 - \frac{-1(-2)(-3)}{1\cdot 2\cdot 3}1^{-4}y^3 + \cdots$$
$$= 1 + y + y^2 + y^3 + \cdots.$$

The $(r+1)$th term is evidently y^r.

Since $(1-y)^{-1} = \frac{1}{1-y}$, the above expansion of $(1-y)^{-1}$ may be verified by division.

16. Expand $(a+x)^{\frac{1}{2}}$ to five terms and find the 10th term.

Expand to four terms:

17. $(1-a)^{-1}$.

18. $(1+a)^{-1}$.

19. $(a-b)^{\frac{1}{2}}$.

20. $\sqrt{4+x}$.

21. $(a+b)^{\frac{2}{3}}$.

22. $\sqrt[4]{(a-b)^3}$.

23. $\sqrt{(9-x)^3}$.

24. $(a+b)^{-\frac{1}{2}}$.

25. $(1+x)^{\frac{2}{5}}$.

26. $(1-x)^{-3}$.

27. $(a^{\frac{2}{3}} - x^{-1})^{\frac{3}{2}}$.

28. $(a^{\frac{1}{2}} - x^{\frac{1}{3}})^{-6}$.

29. Find the square root of 24 to three decimal places.

SOLUTION. $\sqrt{24} = (24)^{\frac{1}{2}} = (25-1)^{\frac{1}{2}} = (25)^{\frac{1}{2}}(1-\frac{1}{25})^{\frac{1}{2}} = 5(1-\frac{1}{25})^{\frac{1}{2}}$

$$= 5\left[1 - \frac{1}{2}\left(\frac{1}{25}\right) + \frac{\frac{1}{2}(-\frac{1}{2})}{1\cdot 2}\left(\frac{1}{25}\right)^2 - \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{1\cdot 2\cdot 3}\left(\frac{1}{25}\right)^3 + \cdots\right]$$

$$= 5 - .1 - .001 - .00002 - \cdots = 4.89898 - = 4.899, \text{ nearly.}$$

Find, to three decimal places, the value of:

30. $\sqrt{5}$.

31. $\sqrt{17}$.

32. $\sqrt{26}$.

33. $\sqrt[3]{25}$.

34. $\sqrt[3]{9}$.

35. $\sqrt[5]{30}$.

LOGARITHMS

558. Early in the seventeenth century a scheme was devised to simplify long computations by representing all real positive numbers as powers of some particular number. The *exponents* of these powers, called *logarithms,* were arranged in tables for convenient reference; and in accordance with the principles of exponents, multiplication was replaced by addition, division by subtraction, involution by a single simple multiplication, and evolution by a single simple division.

Lord Napier, a Scotchman, was the inventor of logarithms and he published the first tables, but to Henry Briggs belongs the honor, next to Napier, for their development. He and Napier independently thought of the advantage of a system that would represent all numbers as powers of 10 to be used with our decimal system of notation, but after consultation with each other and because of Napier's declining health, it was left to Briggs to work out the system that is in common use.

559. The exponent of the power to which a fixed number, called the **base,** must be raised in order to produce a given number is called the **logarithm** of the given number.

When 2 is the base, the logarithm of 8 is 3, for $8 = 2^3$.

When 10 is the base, the logarithm of 100 is 2, for $100 = 10^2$; the logarithm of 1000 is 3, for $1000 = 10^3$; the logarithm of 10,000 is 4, for $10{,}000 = 10^4$.

560. When a is the base, x the exponent, and m the given number, that is, when $a^x = m$, x is the logarithm of the number m to the base a, written $\log_a m = x$.

When the base is 10, it is not indicated. Thus, the logarithm of 100 to *the base 10 is 2. It is* written, $\log 100 = 2$.

561. Logarithms may be computed with any arithmetical number except 1 as a base, but the base of the **common,** or **Briggs, system** of logarithms is 10.

Since $10^0 = 1$, the logarithm of 1 is 0.

Since $10^1 = 10$, the logarithm of 10 is 1.

Since $10^2 = 100$, the logarithm of 100 is 2.

Since $10^3 = 1000$, the logarithm of 1000 is 3.

Since $10^{-1} = \frac{1}{10}$, the logarithm of .1 is -1.

Since $10^{-2} = \frac{1}{100}$, the logarithm of .01 is -2.

562. It is evident, then, that the logarithm of any number between 1 and 10 is a number greater than 0 and less than 1. For example, the logarithm of 4 is approximately 0.6021.

Again, the logarithm of any number between 10 and 100 is a number greater than 1 and less than 2. For example, the logarithm of 50 is approximately 1.6990.

Most logarithms are endless decimals. All the laws established for other exponents apply also to logarithms, but the proofs have been omitted as being too difficult for the beginner.

563. The integral part of a logarithm is called the **characteristic**; the fractional or decimal part, the **mantissa.**

In log 50 = 1.6990, the characteristic is 1 and the mantissa is .6990.

564. The following illustrate *characteristics, mantissas,* and their significance:

$\log 4580 = 3.6609$; that is, $4580 = 10^{3.6609}$.

$\log 458.0 = 2.6609$; that is, $458.0 = 10^{2.6609}$.

$\log 45.80 = 1.6609$; that is, $45.80 = 10^{1.6609}$.

$\log 4.580 = 0.6609$; that is, $4.580 = 10^{0.6609}$.

$\log .4580 = \bar{1}.6609$; that is, $.4580 = 10^{-1+.6609}$.

$\log .0458 = \bar{2}.6609$; that is, $.0458 = 10^{-2+.6609}$.

$\log .00458 = \bar{3}.6609$; that is, $.00458 = 10^{-3+.6609}$.

From the above examples it is evident that:

565. PRINCIPLES.—1. *The characteristic of the logarithm of a number greater than* 1 *is either positive or zero and* 1 *less than the number of digits in the integral part of the number.*

2. *The characteristic of the logarithm of a decimal is negative and numerically* 1 *greater than the number of ciphers immediately following the decimal point.*

566. To avoid writing a negative characteristic before a positive mantissa, it is customary to add 10 or some multiple of 10 to the negative characteristic, and to indicate that the number added is to be subtracted from the whole logarithm.

Thus, $\bar{1}.6609$ is written $9.6609 - 10$; $\bar{2}.3010$ is written $8.3010 - 10$ or sometimes $18.3010 - 20$, $28.3010 - 30$, etc.

567. It is evident, also, from the examples in § 564, that in the logarithms of numbers expressed by the same figures in the same order, the decimal parts, or *mantissas*, are the same, and the logarithms differ only in their *characteristics*. Hence, tables of logarithms contain only the *mantissas*.

568. The table of logarithms on the two following pages gives the decimal parts, or mantissas, to the nearest fourth place, of the common logarithms of all numbers from 1 to 1000.

569. To find the logarithm of a number.

EXERCISES

1. Find the logarithm of 765.

SOLUTION.—In the following table, the letter **N** designates a vertical column of numbers from **10** to **99** inclusive, and also a horizontal row of figures **0**, **1**, **2**, **3**, **4**, **5**, **6**, **7**, **8**, **9**. The first two figures of 765 appear as the number **76** in the vertical column marked **N** on page 427, and the third figure **5** in the horizontal row marked **N**.

In the same horizontal row as **76** are found the mantissas of the logarithms of the numbers 760, 761, 762, 763, 764, 765, etc. The mantissa of the logarithm of 765 is found in this row under **5**, the third figure of 765. It is 8837 and means .8837.

By Prin. 1, the characteristic of the logarithm of 765 is 2.

Hence, the logarithm of 765 is 2.8837.

TABLE OF COMMON LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
N	0	1	2	3	4	5	6	7	8	9

TABLE OF COMMON LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
N	0	1	2	3	4	5	6	7	8	9

2. Find the logarithm of 4.

Solution. — Although the numbers in the table appear to begin with 100, the table really includes all numbers from 1 to 1000, since numbers expressed by less than three figures may be expressed by three figures by adding decimal ciphers. Since $4 = 4.00$, and since, § 567, the mantissa of the logarithm of 4.00 is the same as that of 400, which is .6021, the mantissa of the logarithm of 4 is .6021.

By Prin. 1, the characteristic of the logarithm of 4 is 0.

Therefore, the logarithm of 4 is 0.6021.

Verify the following from the table:

3. $\log 10 = 1.0000.$	9. $\log .2 = 9.3010 - 10.$
4. $\log 100 = 2.0000.$	10. $\log 542 = 2.7340.$
5. $\log 110 = 2.0414.$	11. $\log 345 = 2.5378.$
6. $\log 2 = 0.3010.$	12. $\log 5.07 = 0.7050.$
7. $\log 20 = 1.3010.$	13. $\log 78.5 = 1.8949.$
8. $\log 200 = 2.3010.$	14. $\log .981 = 9.9917 - 10.$

15. Find the logarithm of 6253.

Solution. — Since the table contains the mantissas not only of the logarithms of numbers expressed by three figures, but also of logarithms expressed by four figures when the last figure is 0, the mantissa of the logarithm of 625 is first found, since, § 567, it is the same as the mantissa of the logarithm of 6250. It is found to be .7959.

The next greater mantissa is .7966, the mantissa of the logarithm of 6260. Since the numbers 6250 and 6260 differ by 10, and the mantissas of their logarithms differ by 7 ten-thousandths, it may be assumed as sufficiently accurate that each increase of 1 unit, as 6250 increases to 6260, produces a corresponding increase of .1 of 7 ten-thousandths in the mantissa of the logarithm. Consequently, 3 added to 6250 will add .3 of 7 ten-thousandths, or 2 ten-thousandths, to the mantissa of the logarithm of 6250 for the mantissa of the logarithm of 6253.

Hence, the mantissa of the logarithm of 6253 is $.7959 + .0002$, or .7961.

Since 6253 is an integer of 4 digits, the characteristic is 3 (Prin. 1).

Therefore, the logarithm of 6253 is 3.7961.

Note. — The difference between two successive mantissas in the table *is called the tabular difference*.

Find the logarithm of:

16. 1054.	**20.** 21.09.	**24.** .09095.
17. 1272.	**21.** 3.060.	**25.** .10125.
18. .0165.	**22.** 441.1.	**26.** 54.675.
19. 1906.	**23.** .7854.	**27.** .09885.

570. To find a number whose logarithm is given.

The number that corresponds to a given logarithm is called its **antilogarithm.**

Thus, since the logarithm of 62 is 1.7924, the antilogarithm of 1.7924 is 62.

EXERCISES

571. 1. Find the number whose logarithm is 0.9472.

Solution. — The two mantissas adjacent to the given mantissa are .9469 and .9474, corresponding to the numbers 8.85 and 8.86, since the given characteristic is 0. The given mantissa is 3 ten-thousandths greater than the mantissa of the logarithm of 8.85, and the mantissa of the logarithm of 8.86 is 5 ten-thousandths greater than that of the logarithm of 8.85.

Since the numbers 8.85 and 8.86 differ by 1 one-hundredth, and the mantissas of their logarithms differ by 5 ten-thousandths, it may be assumed as sufficiently accurate that each increase of 1 ten-thousandth in the mantissa is produced by an increase of $\frac{1}{5}$ of 1 one-hundredth in the number. Consequently, an increase of 3 ten-thousandths in the mantissa is produced by an increase of $\frac{3}{5}$ of 1 one-hundredth, or .006, in the number.

Hence, the number whose logarithm is 0.9472 is 8.856.

2. Find the antilogarithm of 9.4180 − 10.

Solution. — Given mantissa,	.4180	
Mantissa next less,	.4166	; figures corresponding, 261.
Difference,	14	
Tabular difference,	17)14(.8	

Hence, the figures corresponding to the given mantissa are 2618.

Since the characteristic is 9 − 10, or − 1, the number is a decimal with no ciphers immediately following the decimal point (Prin. 2).

Hence, the antilogarithm of 9.4180 − 10 is .2618.

Find the antilogarithm of:

3. 0.3010.	8. 3.9545.	13. 9.3685 − 10.
4. 1.6021.	9. 0.8794.	14. 8.9932 − 10.
5. 2.9031.	10. 2.9371.	15. 8.9535 − 10.
6. 1.6669.	11. 0.8294.	16. 7.7168 − 10.
7. 2.7971.	12. 1.9039.	17. 6.7016 − 10.

572. Multiplication by logarithms.

Since logarithms are the exponents of the powers to which a constant number is to be raised, it follows that:

573. PRINCIPLE. — *The logarithm of the product of two or more numbers is equal to the sum of their logarithms;* that is,

To any base, $\log(mn) = \log m + \log n.$

For, let $\log_a m = x$ and $\log_a n = y$, a being any base.

It is to be proved that $\log_a (mn) = x + y.$

§ 559, $a^x = m,$

and $a^y = n.$

Multiplying, § 88, $a^{x+y} = mn.$

Hence, § 560,
$$\log_a (mn) = x + y$$
$$= \log_a m + \log_a n.$$

EXERCISES

574. 1. Multiply .0381 by 77.

SOLUTION

Prin., § 573, $\log(.0381 \times 77) = \log .0381 + \log 77.$

$$\log .0381 = 8.5809 - 10$$
$$\log 77 = 1.8865$$
$$\text{Sum of logs} = 10.4674 - 10$$
$$= 0.4674$$
$$0.4674 = \log 2.934.$$
$$\therefore .0381 \times 77 = 2.934.$$

NOTE. — Three figures of a number corresponding to a logarithm may be found from this table with absolute accuracy, and in most cases the fourth will be correct. In finding logarithms or antilogarithms, allowance should be made for the figures after the fourth, whenever they express .5 or more than .5 of a unit in the fourth place.

Multiply:

2. 3.8 by 56.	6. 2.26 by 85.	10. 289 by .7854.
3. 72 by 39.	7. 7.25 by 240.	11. 42.37 by .236.
4. 8.5 by 6.2.	8. 3272 by 75.	12. 2912 by .7281.
5. 1.64 by 35.	9. .892 by .805.	13. 1.414 by 2.829.

575. Division by logarithms.

Since the logarithms of two numbers to a common base represent exponents of the same number, it follows that:

576. PRINCIPLE. — *The logarithm of the quotient of two numbers is equal to the logarithm of the dividend minus the logarithm of the divisor;* that is,

To any base, $\log (m \div n) = \log m - \log n.$

For, let $\log_a m = x$ and $\log_a n = y$, a being any base.

It is to be proved that $\log_a(m \div n) = x - y.$

§ 559, $a^x = m,$

and $a^y = n.$

Dividing, § 127, $a^{x-y} = m \div n.$

Hence, § 560, $\log_a (m \div n) = x - y$

$= \log_a m - \log_a n.$

EXERCISES

577. 1. Divide .00468 by 73.4.

SOLUTION

Prin., § 576, $\log (.00468 \div 73.4) = \log .00468 - \log 73.4.$

$$\begin{aligned} \log .00468 &= 7.6702 - 10 \\ \log 73.4 &= 1.8657 \\ \hline \text{Difference of logs} &= 5.8045 - 10 \\ 5.8045 - 10 &= \log .00006376. \end{aligned}$$

$\therefore .00468 \div 73.4 = .00006376.$

2. Divide 12.4 by 16.

SOLUTION

Prin., § 576, $\log(12.4 \div 16) = \log 12.4 - \log 16.$

$$\begin{array}{l} \log 12.4 = 1.0934 = 11.0934 - 10 \\ \log 16 \quad = \qquad\qquad 1.2041 \\ \hline \text{Difference of logs} = \; 9.8893 - 10 \end{array}$$

$9.8893 - 10 = \log .775.$

$\therefore\ 12.4 \div 16 = .775.$

SUGGESTION. — The positive part of the logarithm of the dividend may be made to exceed that of the divisor, if necessary to avoid subtracting a larger number from a smaller one as in the above solution, by adding 10 − 10 or 20 − 20, etc.

Divide:

3. 3025 by 55.	8. 10 by 3.14.	13. 1 by 40.
4. 4090 by 32.	9. .6911 by .7854.	14. 1 by 75.
5. 3250 by 57.	10. 2.816 by 22.5.	15. 200 by .5236.
6. .2601 by .68.	11. 4 by .00521.	16. 300 by 17.32.
7. 3950 by .250.	12. 26 by .06771.	17. .220 by .3183.

578. Extended operations in multiplication and division.

Though *negative* numbers have no common logarithms, operations involving negative numbers may be performed by considering only their *absolute values* and then giving to the result the proper sign without regard to the logarithmic work.

Since dividing by a number is equivalent to multiplying by its reciprocal, for every operation of division an operation of multiplication may be substituted. In extended operations in multiplication and division with the aid of logarithms, the latter method of dividing is the more convenient.

579. The logarithm of the reciprocal of a number is called ***the cologarithm*** of the number.

The cologarithm of 100 is the logarithm of $\frac{1}{100}$, which is -2.
It is written, colog $100 = -2$.

580. Since the logarithm of 1 is 0 and the logarithm of a quotient is obtained by subtracting the logarithm of the divisor from that of the dividend, it is evident that the cologarithm of a number is 0 minus the logarithm of the number, or the logarithm of the number with the sign of the logarithm changed; that is, if $\log_a m = x$, then, $\operatorname{colog}_a m = -x$.

Since subtracting a number is equivalent to adding it with its sign changed, it follows that:

581. Principle. — *Instead of subtracting the logarithm of the divisor from that of the dividend, the cologarithm of the divisor may be added to the logarithm of the dividend;* that is,

To any base, $\log(m \div n) = \log m + \operatorname{colog} n.$

EXERCISES

582. 1. Find the value of $\dfrac{.063 \times 58.5 \times 799}{458 \times 15.6 \times .029}$ by logarithms.

Solution

$$\frac{.063 \times 58.5 \times 799}{458 \times 15.6 \times .029} = .063 \times 58.5 \times 799 \times \frac{1}{458} \times \frac{1}{15.6} \times \frac{1}{.029}.$$

$$\begin{aligned}
\log .063 &= 8.7993 - 10\\
\log 58.5 &= 1.7672\\
\log 799 &= 2.9025\\
\operatorname{colog} 458 &= 7.3391 - 10\\
\operatorname{colog} 15.6 &= 8.8069 - 10\\
\operatorname{colog} .029 &= 1.5376\\
\hline
\text{log of result} &= 31.1526 - 30\\
&= 1.1526.
\end{aligned}$$

$$\therefore \text{result} = 14.21.$$

Find the value of:

2. $\dfrac{110 \times 3.1 \times .653}{33 \times 7.854 \times 1.7}$.

3. $\dfrac{15 \times .37 \times 26.16}{11 \times 8 \times .18 \times 6.67}$.

4. $\dfrac{(-3.04) \times .2608}{2.046 \times .06219}$.

5. $\dfrac{600 \times 5 \times 29}{.7854 \times 25000 \times 81.7}$.

6. $\dfrac{3.516 \times 485 \times 65}{3.33 \times 17 \times 18 \times 73}$.

7. $\dfrac{.4051 \times (-12.45)}{(-8.988) \times .01442}$.

8. $\dfrac{78 \times 52 \times 1605}{338 \times 767 \times 431}$.

9. $\dfrac{.5 \times .315 \times 428}{.317 \times .973 \times 43.7}$.

583. Involution by logarithms.

Since logarithms are simply exponents, it follows that:

584. Principle. — *The logarithm of a power of a number is equal to the logarithm of the number multiplied by the index of the power;* that is,

To any base, $\log m^n = n \log m.$

For, let $\log_a m = x$, and let n be any number, a being any base.

It is to be proved that $\log_a m^n = nx$.

§ 559, $a^x = m.$

Raising each member to the nth power, Ax. 6 and § 276, 2,

$$a^{nx} = m^n.$$

Hence, § 560, $\log_a m^n = nx = n \log_a m.$

EXERCISES

585. 1. Find the value of $.25^2$.

Solution

Prin., § 584, $\log .25^2 = 2 \log .25.$

$$\begin{aligned} \log .25 &= 9.3979 - 10. \\ 2 \log .25 &= 18.7958 - 20 \\ &= 8.7958 - 10. \\ 8.7958 - 10 &= \log .06249. \\ \therefore .25^2 &= .06249. \end{aligned}$$

Note. — By actual multiplication it is found that $.25^2 = .0625$, whereas the result obtained by the use of the table is .06249.

Also, by multiplication, $18^2 = 324$, whereas by the use of the table it is *found to be 324.1.* Such inaccuracies must be expected when a four-place *table is* used.

Find by logarithms the value of:

2. 7^2.	7. $.78^2$.	12. 4.07^3.	17. $(\frac{3}{20})^2$.
3. 11^2.	8. 8.05^2.	13. $.543^3$.	18. $(\frac{1}{7})^3$.
4. $(-47)^2$.	9. 8.33^2.	14. $(-7)^4$.	19. $(\frac{128}{1738})^2$.
5. 4.9^2.	10. 6.61^3.	15. 1.02^5.	20. $(\frac{675}{4121})^3$.
6. 5.2^2.	11. $.714^2$.	16. 1.738^3.	21. $(\frac{1}{243})^{.4}$.

586. Evolution by logarithms.

Since logarithms are simply exponents, it follows that:

587. Principle. — *The logarithm of a root of a number is equal to the logarithm of the number divided by the index of the required root;* that is,

To any base, $$\log \sqrt[n]{m} = \frac{\log m}{n}.$$

For, let $\log_a m = x$ and let n be any number, a being any base.

It is to be proved that $\log_a \sqrt[n]{m} = x \div n$.

§ 559, $$a^x = m.$$

Taking the nth root of each member, Ax. 7 and § 290,

$$a^{x \div n} = \sqrt[n]{m}.$$

Hence, § 560, $$\log_a \sqrt[n]{m} = x \div n = \frac{\log_a m}{n}.$$

EXERCISES

588. 1. Find the square root of .1296 by logarithms.

Solution

Prin., § 587, $$\log \sqrt{.1296} = \tfrac{1}{2} \log .1296.$$

$$\log .1296 = 9.1126 - 10.$$

$$2\underline{)19.1126 - 20}$$
$$9.5563 - 10$$

$$9.5563 - 10 = \log .360.$$

$$\therefore \sqrt{.1296} = .36.$$

Find by logarithms the value of:

2. $225^{\frac{1}{2}}$.	8. $(-1331)^{\frac{1}{3}}$.	14. $\sqrt{2}$.	20. $\sqrt[5]{-2}$.
3. $12.25^{\frac{1}{2}}$.	9. $1024^{\frac{7}{10}}$.	15. $\sqrt{3}$.	21. $\sqrt[3]{.027}$.
4. $.2023^{\frac{1}{2}}$.	10. $.6724^{\frac{1}{2}}$.	16. $\sqrt{5}$.	22. $\sqrt{30\frac{4}{5}}$.
5. $326^{\frac{1}{2}}$.	11. $5.929^{\frac{1}{3}}$.	17. $\sqrt{6}$.	23. $\sqrt{.90}$.
6. $.512^{\frac{1}{3}}$.	12. $.4624^{\frac{1}{2}}$.	18. $\sqrt[3]{2}$.	24. $\sqrt{.52}$.
7. $.1182^{\frac{1}{2}}$.	13. $1.4641^{\frac{1}{4}}$.	19. $\sqrt[4]{4}$.	25. $\sqrt[5]{.032}$.

Simplify the following:

26. $\dfrac{176}{15 \times 3.1416}$.

27. $\dfrac{(-100)^2}{48 \times 64 \times 11}$.

28. $\dfrac{52^2 \times 300}{12 \times .31225 \times 400000}$.

29. $\sqrt{\dfrac{400}{55 \times 3.1416}}$.

30. $50 \times \dfrac{2^{3.5}}{8^{1.63}}$.

31. $\dfrac{14.5\sqrt[3]{-1.6}}{11}$.

32. $\sqrt{\dfrac{.434 \times 96^4}{64 \times 1500}}$.

33. $\dfrac{.32 \times 5000 \times 18}{3.14 \times .1222 \times 8}$.

34. $\dfrac{11 \times 2.63 \times 4.263}{48 \times 3.263}$.

35. $\sqrt{\dfrac{3500}{(-1.06)^5}}$.

36. $2^{\frac{1}{2}} \times (\frac{1}{2})^{\frac{2}{3}} \times \sqrt[3]{\frac{3}{2}} \times \sqrt{.1}$.

37. Applying the formula $A = \pi r^2$, find the area (A) of a circle whose radius (r) is 12.35 meters. ($\pi = 3.1416$.)

38. Applying the formula $V = \frac{4}{3}\pi r^3$, find the volume (V) of a sphere whose radius (r) is 40.11 centimeters.

39. The formula $V = .7854\, d^2 l$ gives the volume of a right cylinder d units in diameter and l units long, V, d, and l being *corresponding* units. How many feet of No. 00 wire, which has a diameter of .3648 inches, can be made from a cubic foot of copper?

589. Solution of exponential equations.

Exponential equations, or equations that involve unknown exponents, are solved by the aid of the principle that, in any system, *equal numbers have equal logarithms.*

In simple cases the solution of such equations may be performed by inspection, but in general it is necessary to use a table of logarithms.

EXERCISES

590. 1. Find the value of x in the equation $2^x = 32\sqrt{2}$.

SOLUTION

$$2^x = 32\sqrt{2} = 2^5 2^{\frac{1}{2}} = 2^{\frac{11}{2}};$$

therefore, $\log(2^x) = \log(2^{\frac{11}{2}})$,

or, § 584, $x \log 2 = \frac{11}{2} \log 2.$

Dividing by log 2, $x = \frac{11}{2}.$

2. Find the value of x in the equation $2^x = 48$.

SOLUTION

Taking the logarithm of each member,

$$x \log 2 = \log 48.$$

$$\therefore x = \frac{\log 48}{\log 2}$$

$$= \frac{1.6812}{0.3010} = 5.59^-.$$

3. Solve the equation $3^{2x} - 20 \cdot 3^x + 99 = 0$ for x.

SOLUTION

Factoring the given equation,

$$(3^x - 9)(3^x - 11) = 0.$$

$$\therefore 3^x = 9 \text{ or } 11.$$

Solving the equation $3^x = 9$ by inspection, since $9 = 3^2$,

$$x = 2.$$

Taking the logarithm of each member of $3^x = 11$,

$$x \log 3 = \log 11.$$

$$\therefore x = \frac{\log 11}{\log 3} = \frac{1.0414}{0.4771} = 2.18^+.$$

Therefore, the value of x is either 2 or 2.18^+.

4. Given $x^2 = y^3$ and $x^y = y^x$, to find x and y.

SOLUTION. — Raising the members of the first equation to the xth power, and those of the second equation to the 3d power,

$$x^{2x} = y^{3x},$$

and $$x^{3y} = y^{3x}.$$

Hence, by inspection, $$2x = 3y.$$

Squaring, since $4x^2 = 4y^3$, $$4x^2 = 9y^2 = 4y^3.$$

$$\therefore y = 0 \text{ or } \tfrac{9}{4},$$

and $$x = 0 \text{ or } \tfrac{27}{8}.$$

5. Given $3^x = 2y$ and $2^x = y$, to find x and y.

SOLUTION. $$3^x = 2y. \quad (1)$$

$$2^x = y. \quad (2)$$

Dividing (1) by (2), $$(1.5)^x = 2.$$

$$\therefore x \log 1.5 = \log 2.$$

Hence, by tables, $$x = \frac{\log 2}{\log 1.5} = \frac{0.3010}{0.1761}. \quad (3)$$

By logarithms, $$\log x = 0.2328; \quad (4)$$

whence, by tables, $$x = 1.709. \quad (5)$$

From (2), $$\log y = x \log 2.$$

Then, $$\log \log y = \log x + \log \log 2$$

by (4) and tables, $$= 0.2328 + \bar{1}.4786$$

$$= \bar{1}.7114.$$

Hence, by tables, $$\log y = 0.5145;$$

whence, $$y = 3.270.$$

Solve the following:

6. $3^x = 81.$

7. $4^x = 10.$

8. $2^x = 80.$

9. $3^{x^2} = 9^{2x}.$

10. $2^{3^y} = 512.$

11. $5^{x^6} = 625.$

12. $2^{x^2} = 512.$

13. $(2^x)^2 = 256.$

14. $\begin{cases} 3^x = 2y, \\ 4^x = 20y. \end{cases}$

15. $3^{2x} + 243 = 36 \cdot 3^x.$

16. $\log \log x = \log 2.$

17. $\begin{cases} 2^{x+y} = 6, \\ 2^{x+1} = 3^y. \end{cases}$

18. $\begin{cases} 4^{x+y} = 32, \\ 2^{2x-y} = 4. \end{cases}$

19. $\begin{cases} 2^x = y, \\ x = 1 + \log y. \end{cases}$

591. Logarithms applied to the solution of problems in compound interest and annuities.

Since the amount of any principal at 6 % interest, compounded annually, for 1 year is 1.06 times the principal; for two years, 1.06×1.06, or 1.06^2, times the principal; for 3 years, $1.06 \times 1.06 \times 1.06$, or 1.06^3, times the principal, etc., the amount (A) of any principal (P) for n years at any rate per cent (r) will be

$$A = P(1+r)^n.$$

Expressing this formula by logarithms,

$$\log A = \log P + n \log(1+r). \quad (1)$$

$$\therefore \log P = \log A - n \log(1+r); \quad (2)$$

also $$\log(1+r) = \frac{\log A - \log P}{n}, \quad (3)$$

and $$n = \frac{\log A - \log P}{\log(1+r)}. \quad (4)$$

EXERCISES

592. 1. What is the amount of \$475 for 10 years at 6 % compound interest?

SOLUTION

$$A = P(1+r)^n.$$

$$\begin{array}{ll} \log 475 & = 2.6767 \\ \log 1.06^{10} & = 0.2530 \\ \hline \log A & = 2.9297 \end{array}$$

$$\therefore A = \$850.60.$$

NOTE. — In accordance with the note on page 431, antilogarithms are carried out only to the nearest fourth significant figure.

Find the amount, at compound interest, of:

2. \$225, 5 years, 8 %.
3. \$700, 5 years, 6 %.
4. \$400, 10 years, 3 %.
5. \$1200, 20 years, 4 %.

6. What principal will amount to \$1000 in 10 years at 5 % compound interest?

7. What sum of money invested at 4 % compound interest, payable semiannually, will amount to \$743 in 10 years?

8. What principal loaned at 4 % compound interest will amount to \$1500 in 10 years?

9. What sum invested at 4 % compound interest at a child's birth will amount to \$1000 when he is 21 years old?

10. In what time will \$800 amount to \$1834.50, if put at compound interest at 5 % ?

11. What is the rate per cent when \$300 loaned at compound interest for 6 years amounts to \$402?

12. A man agreed to loan \$1000 at 6 % compound interest for a time long enough for the principal to double itself. How long was the money at interest?

593. A sum of money to be paid periodically for a given number of years, during the life of a person, or forever, is called an **annuity**.

The payments may be made once a year, or twice, or four times a year, etc.

Interest is allowed upon deferred payments.

594. To find the amount of an annuity left unpaid for a given number of years, compound interest being allowed.

An annuity of a dollars per year, payable at the end of each year, will amount to a dollars at the end of the first year. If unpaid and drawing compound interest at a rate r, the accumulation at the end of the second year will be $a + a(1 + r)$ dollars; at the end of the third year, $a + a(1 + r) + a(1 + r)^2$ dollars; and so on.

Let a represent the annuity, n the number of years, r the rate, and A the whole amount due at the end of the nth year.

Then, $A = a + a(1 + r) + a(1 + r)^2 + \cdots + a(1 + r)^{n-1}$

$= a\{1 + (1 + r) + (1 + r)^2 + \cdots + (1 + r)^{n-1}\}.$

Since the terms of A form a geometrical progression in which $1 + r$ is the ratio, § 525, the sum of the series is

$$A = \frac{a}{r}[(1 + r)^n - 1].$$

EXERCISES

595. 1. What will be the amount of annuity of \$100 remaining unpaid for 10 years at 6% compound interest?

SOLUTION

$$A = \frac{a}{r}[(1 + r)^n - 1].$$

$$\begin{aligned} \log 1.06^{10} &= .2530 \\ \therefore 1.06^{10} &= 1.7904 \\ \text{and} \quad 1.06^{10} - 1 &= .7904 \\ \hline \log 100 &= 2.0000 \\ \log .7904 &= 9.8978 - 10 \\ \text{colog } .06 &= 1.2218 \\ \hline \therefore \log A &= 13.1196 - 10 \\ &= 3.1196. \end{aligned}$$

Hence, $A = \$1317$, the amount of the annuity.

2. To what sum will an annuity of \$25 amount in 20 years at 4% compound interest?

3. What will be the amount of an annuity of \$17.76 remaining unpaid for 25 years, at $3\frac{1}{2}$% compound interest?

4. What annuity will amount to \$1000 in 10 years at 5% compound interest?

5. What annuity will amount to \$5000 in 12 years at 3% compound interest?

596. A sum that will amount to the value of an annuity, if put at interest at the given rate for the given time, is called the **present value** of the annuity.

Sometimes annuities, drawing interest, are not payable until after a *certain number of years.*

597. Let P denote the present value of an annuity due in n years, with compound interest at a rate r. Then, the amount of P at the end of the period will be found thus:

By § 591, $$A = P(1+r)^n.$$

But, § 594, $$A = \frac{a}{r}[(1+r)^n - 1].$$

Hence, Ax. 5, $$P(1+r)^n = \frac{a}{r}[(1+r)^n - 1].$$

$$\therefore\ P = \frac{a}{r}\cdot\frac{(1+r)^n - 1}{(1+r)^n}.$$

EXERCISES

598. 1. What is the present value of an annuity of \$100 to continue 10 years at 6 % compound interest?

SOLUTION

$$P = \frac{a}{r}\cdot\frac{(1+r)^n - 1}{(1+r)^n}.$$

$$\begin{aligned}
\log 1.06^{10} &= .2530\\
\therefore\ 1.06^{10} &= 1.7904\\
\text{and}\quad 1.06^{10} - 1 &= .7904\\
\hline
\log 100 &= 2.0000\\
\log .7904 &= 9.8978 - 10\\
\text{colog } .06 &= 1.2218\\
\text{colog } 1.06^{10} &= 9.7470 - 10\\
\hline
\therefore\ \log P &= 22.8666 - 20\\
&= 2.8666.
\end{aligned}$$

Hence, $P =$ \$735.50, the present value.

2. What is the present value of an annuity of \$300 for 5 years at 4 % compound interest?

3. What is the present value of an annuity of \$1000 to continue 20 years, if compound interest at $4\frac{1}{2}$ % is allowed?

4. Find the present value of an annuity of £2000 payable in 10 years, interest being reckoned at 3 %.

PERMUTATIONS AND COMBINATIONS

599. All the different orders in which it is possible to *arrange* a given number of things, by taking either some or all of them at a time, are called the **permutations** of the things.

Thus, the permutations of the letters *a* and *b* are *ab*, *ba*; the permutations of three letters *a*, *b*, and *c*, two at a time, are *ab*, *ac*, *ba*, *bc*, *ca*, *cb*.

600. All the different *selections* that it is possible to make from a given number of things, by taking either some or all of them at a time, without regard to the order in which they are placed, are called the **combinations** of the things.

Thus, while the permutations of three letters, *a*, *b*, and *c*, two at a time, are *ab* and *ba*, *bc* and *cb*, and *ca* and *ac*, their *combinations*, two at a time, are *ab* (or *ba*, but not both), *bc* (or *cb*), and *ac* (or *ca*); again, the six permutations of these three letters among themselves, viz., *abc*, *acb*, *bca*, *bac*, *cab*, and *cba*, form but one combination, *abc* (or *acb*, or *bca*, or *bac*, or *cab*, or *cba*).

It is evident that there can be only one combination of any number of things taken all at a time.

601. Notation. — The *symbol* for the number of *permutations* of n different things, taken r at a time, is P^n_r; of n different things, taken n at a time, or all together, is P^n_n.

Instead of P^n_r, sometimes ${}^nP_r$, ${}_nP_r$, or $P_{n,r}$ is used. Similarly, for P^n_n, sometimes ${}^nP_n$, ${}_nP_n$, or $P_{n,n}$ is used.

The *symbol* for the number of *combinations* of n different things, taken r at a time, is C^n_r; of n different things, taken n at a time, or all together, is C^n_n.

Instead of C^n_r, sometimes ${}^nC_r$, ${}_nC_r$, or $C_{n,r}$ is used. Similarly, for C^n_n, sometimes ${}^nC_n$, ${}_nC_n$, or $C_{n,n}$ is used.

602. The product of the successive integers from 1 to n, or from n to 1, inclusive, is called **factorial n,** written $\underline{|n}$, or $n!$.

$$\underline{|5} = 1 \times 2 \times 3 \times 4 \times 5, \text{ or } 5 \times 4 \times 3 \times 2 \times 1;$$

$$\underline{|n} = 1 \cdot 2 \cdot 3 \cdots (n-2)(n-1)n, \text{ or } n(n-1)(n-2)(n-3) \cdots 3 \cdot 2 \cdot 1.$$

603. To find the number of permutations of *n* different things taken *r* at a time.

Since the permutations of a, b, and c, taken 2 at a time, are ab and ac, ba and bc, ca and cb, formed by writing after each of the letters, a, b, and c, each of the other letters in turn, the number of permutations of 3 different things taken 2 at a time is 3×2.

The number of permutations of n letters taken 2 at a time may be found by associating with each of the n letters each of the $n-1$ other letters. Consequently, the number of permutations of n different things taken 2 at a time is $n(n-1)$.

Since the number of permutations of n letters 2 at a time is $n(n-1)$, if the letters are taken 3 at a time there will be $n-2$ letters, each of which may be associated with each of the $n(n-1)$ permutations of letters taken 2 at a time. Hence, the number of permutations of n different things taken 3 at a time is

$$n(n-1)(n-2).$$

PRINCIPLE 1. — *The number of permutations of n different things taken r at a time is equal to the continued product of the natural numbers from n to n —(r — 1) inclusive. The number of factors is r.* That is,

$$\begin{aligned} P_r^n &= n(n-1)(n-2) \cdots \text{to } r \text{ factors} \\ &= n(n-1)(n-2) \cdots (n-r+1). \end{aligned} \qquad \text{(I)}$$

Multiplying and dividing the second member of (I) by $(n-r)(n-r-1)(n-r-2) \cdots 2 \cdot 1$; that is, by $\underline{|n-r}$,

$$P_r^n = \frac{\underline{|n}}{\underline{|n-r}}. \qquad \text{(II)}$$

NOTE. — *It* will usually be more convenient to employ formula (I) in solving numerical exercises; but when literal results are desired, formula (II) will be preferable.

604. When $r = n$, that is, when the things are taken all together, the last, or nth, factor in (I) is 1. Consequently,

PRINCIPLE 2. — *The number of permutations of n different things taken all at a time is equal to* $\underline{|n}$. That is,

$$P_n^n = n(n-1)(n-2)\cdots 2\cdot 1 = \underline{|n}. \qquad \text{(III)}$$

EXERCISES

605. 1. Three boys enter a car in which there are 5 empty seats. In how many ways may they choose seats?

SOLUTION. — Since the first boy may choose any one of 5 seats, and since for each seat that he may choose the second boy may choose any one of the 4 seats remaining, the greatest possible number of ways in which two of the boys may be seated is 5×4.

Again, since after each choice of seats made by two of the boys there will be left to the third boy a choice of one of the 3 seats remaining, the number of ways in which all may choose seats is $5 \times 4 \times 3$, or 60.

Or, by (I), $\quad P_r^n = n(n-1)(n-2)\cdots(n-r+1)$;

that is, $\quad P_3^5 = 5 \times 4 \times 3 = 60.$

2. How many numbers between 100 and 1000 can be expressed by the figures 1, 3, 5?

SOLUTION. — Since the numbers lie between 100 and 1000, each must be expressed by three figures. Hence, the number of numbers between 100 and 1000 that can be expressed by the figures 1, 3, and 5 is the same as the number of permutations of these 3 figures taken 3 at a time.

Since, Prin. 2, $\quad P_3^3 = \underline{|3} = 3\cdot 2\cdot 1 = 6,$

there are six such numbers. They are 135, 153, 351, 315, 513, and 531.

3. How many permutations can be made of the letters in the word *Albany*, each beginning with capital A?

SOLUTION. — Since A is to be prefixed to each permutation of the 5 other letters, the required number is

$$P_5^5 = 5 \times 4 \times 3 \times 2 \times 1 = 120.$$

4. In how many orders may 4 persons sit on a bench?

5. How many permutations may be made of the letters in the word *number*?

6. If 10 athletes run a race, in how many ways may the first and second prizes be awarded?

7. In how many different orders may the colors violet, indigo, blue, green, yellow, orange, and red be arranged?

8. There are 5 routes to the top of a mountain. In how many ways may a person go up and return by a different way?

606. To find the number of combinations of *n* different things taken *r* at a time.

Since two letters, as a and b, have two permutations, ab and ba, but form only one combination, the number of combinations of n letters taken 2 at a time is one half the number of permutations of n letters taken 2 at a time.

Since three letters taken 3 at a time have 3×2 permutations, but form only one combination, the number of combinations of n letters taken 3 at a time is obtained by dividing the number of permutations of n letters taken 3 at a time by 3×2.

Since four letters taken 4 at a time have $\lfloor\underline{4}$ permutations but form only one combination, to obtain the number of combinations of n letters taken 4 at a time, the number of permutations of n letters taken 4 at a time must be divided by $\lfloor\underline{4}$. Hence,

PRINCIPLE 3. — *The number of combinations of n different things taken r at a time is equal to the number of permutations of n different things taken r at a time, divided by the number of permutations of r different things taken all together.* That is,

$$C^n_r = P^n_r \div P^r_r = \frac{n(n-1)(n-2) \cdots \text{to } r \text{ factors}}{r(r-1)(r-2) \cdots \text{to } r \text{ factors}}$$

$$= \frac{n(n-1)(n-2) \cdots (n-r+1)}{1 \cdot 2 \cdot 3 \cdots r}. \qquad \text{(IV)}$$

Or, by (II) and (III),

$$C^n_r = P^n_r \div P^r_r = \frac{\lfloor\underline{n}}{\lfloor\underline{n-r}} \div \lfloor\underline{r}$$

$$= \frac{\lfloor\underline{n}}{\lfloor\underline{r}\,\lfloor\underline{n-r}}. \qquad \text{(V)}$$

607. Since for every combination of r things out of n different things there is left a combination of $n-r$ things, it follows that:

Principle 4. — *The number of combinations of n different things is the same when taken $n-r$ at a time as when taken r at a time.* That is,

$$C^n_{n-r} = C^n_r = \frac{\lfloor n}{\lfloor r \lfloor n-r}. \qquad \text{(VI)}$$

The above principle may be established as follows:

By (V),
$$C^n_r = \frac{\lfloor n}{\lfloor r \lfloor n-r}. \qquad (1)$$

Substituting $n-r$ for r,
$$C^n_{n-r} = \frac{\lfloor n}{\lfloor n-r \lfloor n-(n-r)}$$

$$= \frac{\lfloor n}{\lfloor n-r \lfloor r}. \qquad (2)$$

Since the second members of (1) and (2) are identical, $C^n_{n-r} = C^n_r$.

The above principle is useful in abridging numerical computations.

Thus, the number of combinations of 18 things taken 16 at a time is computed by Prin. 3 as follows:

$$C^{18}_{16} = \frac{18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12 \cdot 13 \cdot 14 \cdot 15 \cdot 16} = 153.$$

But by Prin. 4, the computation is abridged as follows:

$$C^{18}_{16} = C^{18}_2 = \frac{18 \cdot 17}{1 \cdot 2} = 153.$$

EXERCISES

608. 1. A man has 6 friends and wishes to invite 4 of them to dinner. In how many ways may he select his guests?

Solution. — Since each party, or combination, of 4 guests could be arranged, or permuted, in $\lfloor 4$ ways, the number of combinations must be $\frac{1}{\lfloor 4}$ of the number of permutations of 6 things taken 4 at a time.

Hence, the number of ways is

$$C^6_4 = P^6_4 \div P^4_4 = \frac{6 \times 5 \times 4 \times 3}{1 \times 2 \times 3 \times 4} = 15.$$

2. A man and his wife wish to invite 11 of their friends, 6 men and 5 women, to dinner, but find that they can entertain only 8 guests. In how many ways may they invite 4 men and 4 women?

SOLUTION. — As in the previous exercise, 4 men may be selected from 6 men in 15 ways, and in a similar manner 4 women may be selected from 5 women in 5 ways.

Since, when any set of 4 men has been invited, the party of 8 may be completed by inviting any one of 5 sets of 4 women, the whole number of different parties that it is possible to invite is 15×5, or 75. That is,

$$C_4^6 \times C_4^5 = \frac{6 \cdot 5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3 \cdot 4} \times \frac{5 \cdot 4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3 \cdot 4} = 75.$$

3. In how many ways may a baseball nine be selected from 12 candidates?

4. How many different combinations of 5 cards may be formed from 52 cards?

5. Which is the greater, C_8^{10} or C_3^{10}? C_4^{10} or C_5^{10}? C_8^{12} or C_9^{12}?

6. From 11 Republicans and 10 Democrats how many different committees may be selected composed of 6 Republicans and 5 Democrats?

7. A man forgets the combination of figures and letters by which his safe is opened. They are arranged on the circumferences of three wheels, one bearing the numbers 0 to 9 inclusive, another the letters A to M inclusive, and the third the letters N to Z inclusive. What is the greatest number of trials he may have to make to open the safe?

8. From 6 consonants and 4 vowels how many words may be formed each consisting of 4 consonants and 2 vowels, if any arrangement of the letters is considered a word?

SUGGESTION. — The number of combinations is $C_4^6 \times C_2^4$; and since by permuting the letters of each combination $\underline{|6}$ words can be formed, the number of words is $C_4^6 \times C_2^4 \times \underline{|6}$.

9. *In an* omnibus that will seat 8 persons on a side there *are seated* 4 persons, 3 on one side and 1 on the other. In how *many* ways may 12 more persons be seated?

SOLUTION. — Since 5 persons must take seats on one side and 7 persons on the other, 12 persons are to be divided into two classes, 5 and 7. The number of these combinations, formula (V), is

$$C_5^{12}, \text{ or } C_7^{12}, = \frac{\lfloor\underline{12}}{\lfloor\underline{5}\,\lfloor\underline{7}}.$$

Since each combination of 5 may have $\lfloor\underline{5}$ permutations of the 5 that compose it, and each combination of 7 may have $\lfloor\underline{7}$ permutations each of which may be associated with each of the $\lfloor\underline{5}$ permutations, the required number of ways is $C_5^{12} \times P_5^5 \times P_7^7$; that is,

$$\frac{\lfloor\underline{12}}{\lfloor\underline{5}\,\lfloor\underline{7}} \times \lfloor\underline{5} \times \lfloor\underline{7} = \lfloor\underline{12}.$$

Or, 12 persons may be seated in 12 seats in $P_{12}^{12} = \lfloor\underline{12}$ ways.

10. Out of 20 consonants and 5 vowels how many words containing 3 consonants and 3 vowels may be formed, if any arrangement of the letters is considered a word?

11. How many different sums may be paid with a cent, a 5-cent piece, a dime, a quarter, and a dollar?

12. From 5 boys and 5 girls how many committees of 6 may be selected so as to contain at least 2 boys?

13. If $C_5^n = 2\,C_2^n$, find the number of things.

SOLUTION. — By formula (V), $C_5^n = \dfrac{\lfloor\underline{n}}{\lfloor\underline{5}\,\lfloor\underline{n-5}}$ and $C_2^n = \dfrac{\lfloor\underline{n}}{\lfloor\underline{2}\,\lfloor\underline{n-2}}$.

Since $C_5^n = 2\,C_2^n$,

$$\frac{\lfloor\underline{n}}{\lfloor\underline{5}\,\lfloor\underline{n-5}} = \frac{2\lfloor\underline{n}}{\lfloor\underline{2}\,\lfloor\underline{n-2}}.$$

$$\frac{1}{\lfloor\underline{5}\,\lfloor\underline{n-5}} = \frac{1}{\lfloor\underline{n-2}}.$$

$$\therefore \quad \lfloor\underline{n-2} = \lfloor\underline{5}\,\lfloor\underline{n-5}.$$

$$\frac{\lfloor\underline{n-2}}{\lfloor\underline{n-5}} = \lfloor\underline{5} = 5 \times 4 \times 3 \times 2 \times 1;$$

that is, $$(n-2)(n-3)(n-4) = 6 \times 5 \times 4.$$

$$\therefore n = 8.$$

14. *If* $3\,C_3^n = 2\,C_4^{n+1}$, find n, C_3^n, and C_4^{n+1}.

609. To find the number of permutations of n things taken n at a time when they are not all different.

If, in the permutation (a, b, c, d, e, f, g), the letters b, d, and g are permuted while the other letters remain fixed in position, the resulting number of permutations will be the same as the number of permutations of b, d, and g. If b, d, and g are different things, the number of permutations resulting will be $\underline{|3}$; but if b, d, and g become alike, there will be but 1 permutation.

That is, the number of permutations of any number of things, 3 of which are alike, is equal to the number of permutations of the things, considered as all different, divided by $\underline{|3}$; if 4 of the things are alike, by $\underline{|4}$; if p of the things are alike, by $\underline{|p}$.

Hence, it follows that:

PRINCIPLE 5. — *The number of permutations of n things, taken all together, when p of them are alike, is* $\dfrac{\underline{|n}}{\underline{|p}}$.

If q of the remaining $n-p$ different things become alike, but different from the p like things, the number of permutations must be divided by $\underline{|q}$; if r others become alike, by $\underline{|r}$; etc. Hence, it follows that:

PRINCIPLE 6. — *The number of permutations of n things, taken all together, when p of them are of one kind, q of another, r of another, etc., is* $\dfrac{\underline{|n}}{\underline{|p}\,\underline{|q}\,\underline{|r}\cdots}$.

EXERCISES

610. 1. How many permutations may be made with the letters of the word *Mississippi* taken all together?

SOLUTION. — The number is $\dfrac{\underline{|11}}{\underline{|4}\,\underline{|4}\,\underline{|2}} = 34650$.

2. How many permutations may be made with the letters of each of the following words, taken all at a time in each case: *characteristic, coefficient, ecclesiastical, divisibility?*

3. How many permutations may be made with the letters represented in the product $a^4b^5c^2$ written out in full?

611. To find the total number of combinations of n different things.

The number of combinations of n different things taken successively 1, 2, 3, $\cdots$ n at a time is called the *total* number of combinations of n things.

The total number of combinations of 2 things is

$$C_1^2 + C_2^2 = 2 + 1 = 3, \text{ or } 2^2 - 1.$$

The total number of combinations of 3 things is

$$C_1^3 + C_2^3 + C_3^3 = 3 + 3 + 1 = 7, \text{ or } 2^3 - 1.$$

The total number of combinations of 4 things is

$$C_1^4 + C_2^4 + C_3^4 + C_4^4 = 4 + 6 + 4 + 1 = 15, \text{ or } 2^4 - 1.$$

Hence, it may be inferred that:

PRINCIPLE 7. — *The total number of combinations of n different things is* $2^n - 1$.

The above principle may be established as follows:

By § 552, when n is a positive integer,

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1 \cdot 2}x^2 + \cdots + \frac{n(n-1)(n-2)\cdots 1}{1 \cdot 2 \cdot 3 \cdots n}x^n.$$

If $x = 1$, $\quad 2^n = 1 + n + \frac{n(n-1)}{1 \cdot 2} + \cdots + \frac{n(n-1)(n-2)\cdots 1}{1 \cdot 2 \cdot 3 \cdots n}$

Prin. 3, $\quad = 1 + C_1^n + C_2^n + \cdots + C_n^n = 1 + C_{\text{total}}^n.$

$$\therefore C_{\text{total}}^n = 2^n - 1.$$

EXERCISES

612. 1. How many different sums may be paid with a cent, a 5-cent piece, a dime, a quarter, a half-dollar, and a dollar?

SOLUTION. $\quad C_{\text{total}}^6 = 2^6 - 1 = 63.$

2. A man has 10 friends. In how many ways may he invite one or more of them to dinner?

3. How many different quantities may be weighed by weights of 1 oz., 1 lb., $\frac{1}{2}$ lb., 5 lb., and 10 lb.?

613. To find for what value of r the number of combinations of n things taken r at a time is greatest.

Since formula (IV), namely,

$$C_r^n = \frac{n(n-1)(n-2)\cdots(n-r+1)}{1\cdot 2\cdot 3\cdots r}, \tag{1}$$

has r factors in the numerator and r factors in the denominator, it may be written,

$$C_r^n = \frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\cdots\frac{n-r+1}{r}. \tag{2}$$

The numerators in these factors begin with n and *decrease* by 1 while the denominators begin with 1 and *increase* by 1. The factors, then, are at first improper fractions and at some point they begin to be proper fractions. Hence, C_r^n is greatest when it is the product of all the fractions that are greater than 1.

1. *When n is an even number.*

In this case, the numerator of the first fraction in (2) is even and the denominator odd, in the second the numerator is odd and the denominator even, and so on alternately; hence, the fraction greater than 1, but nearest to 1, is the fraction whose numerator is 1 greater than its denominator; that is, the value of r must be such that

$$n-r+1=r+1, \text{ or } r=\frac{n}{2}.$$

2. *When n is an odd number.*

In this case, the numerator and denominator of the first fraction in (2) are both odd, of the second both even, and so on alternately; hence, the fraction greater than 1, but nearest to 1, is the fraction whose numerator is 2 greater than its denominator; that is, the value of r must be such that

$$n-r+1=r+2, \text{ or } r=\frac{n-1}{2}.$$

Note. — When n is *odd*, since (IV) $C_r^n = C_{n-r}^n$, there are two values f r for which C_r^n is greatest, the other value being $r = \frac{n+1}{2}$.

EXERCISES

614. 1. For what value of r is C_r^{10} greatest? Find C_r^{10} for that value of r.

2. What is the greatest value of C_r^9?

Solve the following miscellaneous exercises:

3. By permuting the letters of the word *counter*, how many permutations may be formed

(*a*) ending in *er*?

(*b*) with n as the middle letter?

(*c*) without changing the position of any vowel?

(*d*) beginning with a consonant?

(*e*) keeping the vowels in their present order?

Suggestion. — Since in (*e*) the vowels are to be kept in the order o, u, e, the first consonant may be placed successively before o, between o and u, between u and e, and after e; that is, it may be placed in 4 ways. Then the second consonant may be placed in 5 ways, etc.

Or, since the vowels are not interchangeable, they may be considered alike, and principle 6 may be applied.

4. A man has five coats, six vests, and eight pairs of trousers. In how many different suits may he appear?

5. How many signals may be made with 7 flags of different colors displayed either singly, or any number at a time arranged vertically with equal spaces between them?

6. How many permutations of 6 letters may be formed with 3 consonants and 3 vowels, if the vowels are always given the even places?

7. How many numbers may be formed with the digits 1, 2, 3, 4, 3, 2, 1, so that the odd digits always occupy the odd places?

8. If the number of permutations of n different things taken 5 at a time is equal to 24 times the number of permutations of *the same number* of things taken 2 at a time, find n.

COMPLEX NUMBERS

615. The student has learned that the indicated even root of a negative number is called an imaginary number, and that operations involving such numbers are subject to the condition that

$$(\sqrt{-1})^2, \text{ or } i^2, \text{ equals } -1, \text{ not } +1.$$

616. Including all intermediate fractional and surd values, the scale of real numbers may be written

$$\cdots -3 \cdots -2 \cdots -1 \cdots 0 \cdots +1 \cdots +2 \cdots +3 \cdots, \qquad (1)$$

and the scale of imaginary numbers, composed of real multiples of $+i$ and $-i$, may be written

$$\cdots -3i \cdots -2i \cdots -i \cdots 0 \cdots +i \cdots +2i \cdots +3i \cdots. \qquad (2)$$

Since the square of every real number except 0 is positive and the square of every imaginary number except $0i$, or 0, is negative, the scales (1) and (2) have no number in common except 0. Hence,

An imaginary number cannot be equal to a real number nor cancel any part of a real number.

617. The algebraic sum of a real number and an imaginary number is called a **complex number**.

$2 + 3\sqrt{-1}$, or $2 + 3i$, and $a + b\sqrt{-1}$, or $a + bi$, are complex numbers. $a^2 + 2ab\sqrt{-1} - b^2$ is a complex number, since $a^2 + 2ab\sqrt{-1} - b^2 = (a^2 - b^2) + 2ab\sqrt{-1}$.

618. Two complex numbers that differ only in the signs of *their imaginary* terms are called **conjugate complex numbers.**

$a + b\sqrt{-1}$ and $a - b\sqrt{-1}$, or $a + bi$ and $a - bi$, are conjugate complex numbers.

619. Operations involving complex numbers.

EXERCISES

1. Add $3-2\sqrt{-1}$ and $2+5\sqrt{-1}$.

SOLUTION

Since, § 616, the imaginary terms cannot unite with the real terms, the simplest form of the sum is obtained by uniting the real and the imaginary terms separately and indicating the algebraic sum of the results.

$$3-2\sqrt{-1}+2+5\sqrt{-1}=(3+2)+(-2\sqrt{-1}+5\sqrt{-1})$$
$$=5+3\sqrt{-1}.$$

Simplify the following:

2. $(5+\sqrt{-4})+(\sqrt{-9}-3)$.

3. $(2-\sqrt{-16})+(3+\sqrt{-4})$.

4. $(3-\sqrt{-8})+(4+\sqrt{-18})$.

5. $(\sqrt{-20}-\sqrt{16})+(\sqrt{-45}+\sqrt{4})$.

6. $(4+\sqrt{-25})-(2+\sqrt{-4})$.

7. $(3-2\sqrt{-5})-(2-3\sqrt{-5})$.

8. $(2-2\sqrt{-1}+3)-(\sqrt{16}-\sqrt{-16})$.

9. $\sqrt{-49}-2-3\sqrt{-4}-\sqrt{-1}+6$.

10. Expand $(a+b\sqrt{-1})(a+b\sqrt{-1})$.

SOLUTION

§ 105, $(a+b\sqrt{-1})(a+b\sqrt{-1})=a^2+2ab\sqrt{-1}+(b\sqrt{-1})^2$

§ 615, $=a^2+2ab\sqrt{-1}-b^2.$

11. Expand $(\sqrt{5}-\sqrt{-3})^2$.

SOLUTION

$$(\sqrt{5}-\sqrt{-3})^2=5-2\sqrt{-15}+(-3)$$
$$=(5-3)-2\sqrt{-15}$$
$$=2-2\sqrt{-15}.$$

Expand:

12. $(2+3\sqrt{-1})(1+\sqrt{-1})$.

13. $(5-\sqrt{-1})(1-2\sqrt{-1})$.

14. $(\sqrt{2}+\sqrt{-2})(\sqrt{8}-\sqrt{-8})$.

15. $(2+3i)^2$.

16. $(2-3i)^2$.

17. $(a-bi)^2$.

Show that:

18. $(1+\sqrt{-3})(1+\sqrt{-3})(1+\sqrt{-3})=-8$.

19. $(-1+\sqrt{-3})(-1+\sqrt{-3})(-1+\sqrt{-3})=8$.

20. $(-\frac{1}{2}+\frac{1}{2}\sqrt{-3})(-\frac{1}{2}+\frac{1}{2}\sqrt{-3})(-\frac{1}{2}+\frac{1}{2}\sqrt{-3})=1$.

21. Divide $8+\sqrt{-1}$ by $3+2\sqrt{-1}$.

FIRST SOLUTION

$$
\begin{array}{r|l}
8+\sqrt{-1}=6+\sqrt{-1}+2 & 3+2\sqrt{-1} \\ \cline{2-2}
\underline{6+4\sqrt{-1}} & 2-\sqrt{-1} \\
-3\sqrt{-1}+2 & \\
\underline{-3\sqrt{-1}+2} &
\end{array}
$$

The real term of the dividend may always be separated, as above, into two parts, one of which will exactly contain the real term of the divisor.

SECOND SOLUTION

$$\frac{8+\sqrt{-1}}{3+2\sqrt{-1}}=\frac{(8+\sqrt{-1})(3-2\sqrt{-1})}{(3+2\sqrt{-1})(3-2\sqrt{-1})}=\frac{26-13\sqrt{-1}}{9+4}=2-\sqrt{-1}.$$

Divide:

22. 3 by $1-\sqrt{-2}$.

23. 2 by $1+\sqrt{-1}$.

24. $4+\sqrt{4}$ by $2-\sqrt{-2}$.

25. a^2+b^2 by $a-b\sqrt{-1}$.

26. $a-bi$ by $ai+b$.

27. $(1+i)^2$ by $1-i$.

28. Find by inspection the square root of $3+2\sqrt{-10}$.

SOLUTION

$$3+2\sqrt{-10}=(5-2)+2\sqrt{5\cdot -2}=5+2\sqrt{5\cdot -2}+(-2).$$

$$\therefore \sqrt{3+2\sqrt{-10}}=\sqrt{5+2\sqrt{5\cdot -2}+(-2)}=\sqrt{5}+\sqrt{-2}.$$

Find by inspection the square root of:

29. $4+2\sqrt{-21}$.

30. $1+2\sqrt{-6}$.

31. $6-2\sqrt{-7}$.

32. $9+2\sqrt{-22}$.

33. $4\sqrt{-3}-1$.

34. $12\sqrt{-1}-5$.

35. $24\sqrt{-1}-7$.

36. $b^2+2ab\sqrt{-1}-a^2$.

37. Verify that $-1+\sqrt{-1}$ and $-1-\sqrt{-1}$ are roots the equation $x^2+2x+2=0$.

38. Expand $(\frac{1}{2}+\frac{1}{2}\sqrt{-3})^3$.

620. *The sum and the product of two conjugate complex nu bers are both real.*

For, let $a+b\sqrt{-1}$ and $a-b\sqrt{-1}$ be conjugate complex numb Their sum is $2a$.

Since $(\sqrt{-1})^2=-1$, their product is,

§ 114,
$$a^2-(b\sqrt{-1})^2=a^2-(-b^2)$$
$$=a^2+b^2.$$

621. *If two complex numbers are equal, their real parts equal and also their imaginary parts.*

For, let $$a+b\sqrt{-1}=x+y\sqrt{-1}.$$

Then, $$a-x=(y-b)\sqrt{-1},$$

which, § 616, is impossible unless $a=x$ and $y=b$.

622. *If $a+b\sqrt{-1}=0$, a and b being real, then $a=0$ a $b=0$.*

For, if $$a+b\sqrt{-1}=0,$$

then, $$b\sqrt{-1}=-a,$$

and, squaring, $$-b^2=a^2;$$

whence, $$a^2+b^2=0.$$

Now, a^2 and b^2 are both positive; hence, their sum cannot be *less each is separately* 0; that is, $a=0$ and $b=0$.

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